Gibbs Partitions (EPPF's) Derived From a Stable Subordinator are Fox $H-$ and Meijer $G-$Transforms

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Abstract: This paper derives explicit results for the infinite Gibbs partitions generated by the jumps of an $\alpha-$stable subordinator, derived in Pitman (39; 40). We first show that for general α the conditional EPPF can be represented as ratios of Fox H-functions, and in the case of rational α, Meijer G-functions. This extends results for the known case of $\alpha = 1/2$, which can be expressed in terms of Hermite functions, hence answering an open question. Furthermore the results show that the resulting unconditional EPPF's can be expressed in terms of H- and G-transforms indexed by a function h. Hence when h is itself a H- or G-function the EPPF is also an H- or G-function. An implication, in the case of rational α, is that one can compute explicitly thousands of EPPF's derived from possibly exotic special functions. This would also apply to all α except that computations for general Fox H-functions are not yet available. However, moving away from special functions, we demonstrate how results from probability theory may be used to obtain calculations. We show that a forward recursion can be applied that only requires calculation of the simplest components. Additionally we identify general classes of EPPF's where explicit calculations can be carried out using distribution theory. Specifically what we call the Lamperti class and Beta-Gamma class. As a special application, we use the latter class to obtain EPPF's based on mixing distributions derived from

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the laws of of ranked functionals of self similar Markovian excursions. The work serves importantly a dual purpose. One is to obtain explicit calculations for large classes of EPPF's and hence of use to the growing number of applications involving combinatorial stochastic processes. The other, perhaps surprising, is the introduction of new techniques for calculating explicitly certain Fox H-transforms and related quantities via probabilistic arguments.

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1. Introduction

Let S_α, for $0 < \alpha < 1$, denote a positive α-stable random variable, whose law is specified by the Laplace transform,

$$\mathbb{E}[\mathrm{e}^{-\lambda S_\alpha}] = \mathrm{e}^{-\lambda^\alpha}$$

for $\lambda > 0$, and with density denoted as f_α. Then, following (35; 42) it is well-known that

$$S_\alpha = \sum_{i=1}^{\infty} J_i$$

where $J_1 \geq J_2 \geq \ldots > 0$ are the ranked jump sizes of a stable subordinator

$$S_\alpha(s) = \sum_{i=1}^{\infty} J_i \mathbb{I}(U_i \leq s), \quad 0 \leq s \leq 1, \text{ with } S_\alpha(1) = S_\alpha,$$

where the (U_i) are independent random times distributed uniformly on $(0, 1)$ and also independent of the (J_i). Furthermore, the subordinator is characterized by its Lévy density,

$$\rho_\alpha(s) = \frac{\alpha}{\Gamma(1-\alpha)} s^{-\alpha-1} \text{ for } s > 0.$$

Following Kingman (21) and Perman-Pitman-Yor (41; 35; 42), Pitman (39) discussed the laws of the ranked jumps normalized by their random total mass S_α and further conditioned on $S_\alpha = t$, that is,

$$\mathcal{L}\left(P_1, P_2, \ldots \mid S_\alpha = t\right)$$

where $(P_i) = (J_i/S_\alpha)$. The distribution is referred to as the (conditional) *Poisson Kingman distribution with Lévy density* ρ_α and denoted as $\mathrm{PK}(\rho_\alpha|t)$. Furthermore, one can create an infinite number of laws from this construction by mixing over t with any distribution γ on $(0, \infty)$. The law of the sequence (P_i) is then referred to as the *Poisson Kingman distribution with Lévy density ρ_α and mixing distribution* γ, denoted as

$$\mathrm{PK}(\rho_\alpha, \gamma) = \int_0^\infty \mathrm{PK}(\rho_\alpha|t)\gamma(dt).$$

In this paper we provide explicit calculations and interpretations for the *exchangeable partition probability function* (EPPF) which characterizes the law of the exchangeable random partitions $\Pi_\infty = (\Pi_n)$ on $\mathbb{N}$ generated by the ranked jumps of an α stable subordinator, that is, the $\text{PK}(\rho_\alpha, \gamma)$ partitions. Equivalently, using *Kingman's paintbox representation* (38; 12; 10; 11) this class of exchangeable random partitions can all be constructed by a random closed set $Z \subset [0, 1]$ where Z is the scaled range of an $\alpha-$stable subordinator conditioned on its value at a fixed time. Conditional on t this construction produces the $\text{PK}(\rho_\alpha|t)$ partition, where the conditional EPPF was derived by Pitman (39).

Our work may be divided into two parts. One is the use of special functions and the theory of fractional calculus to help interpret these EPPF's and in many cases to obtain explicit numerical calculations. The other is to use some interesting probability distribution theory to obtain explicit results for large classes of EPPF's and which in turn yields calculations for various special functions.

Specifically we will show that for general α these EPPF's may be represented in terms of Fox H-functions, and for the case where α takes on rational values in terms of Meijer G-functions. There are several significant implications of these representations. One is that a large number of special functions commonly appearing in, for instance, physics, probability, finance or fractional calculus, can be represented in terms of H- and G-functions. See, for instance, (13; 20; 22; 27; 26; 28; 45). In addition, because these functions are well understood this offers additional interpretability of the relevant EPPF's. That is one is not merely applying a numerical calculation. The case of rational α is particularly interesting as calculations involving general Meijer G-functions are at the heart of mathematical computer packages such as *Mathematica* and *Maple*. This literally allows one to explicitly calculate thousands of EPPF's, while in the present literature only a few cases of explicit EPPF's are known. However, as of yet, while in the general case of α we can express many EPPF's in terms of H-transforms there does not exist general mathematical packages to compute them. This brings up our other approach which is based on probability distribution theory associated with beta, gamma and stable random variables relying in large part on a recent work of James (18), see also ((17)), and which further relies on some results in Perman-Pitman-Yor (41; 35; 42) and Pitman (40), and some lesser known results for S_α. In this regard, the fact that the relevant components in the EPPF satisfy a forward recursion, which follows from a backward equation as seen in Gnedin and Pitman (12), plays an important role. As will be discussed in Section 6, the recursion shows that one need only calculate the simplest components to calculate all components via a recursion. Specifically, it suffices to compute the probability of having one block in a partition of integers $\{1, \ldots, n\}$, that is,

$$\mathbb{P}(\{1, \ldots, n\}) = V_{n,1}[1-\alpha]_{n-1},$$

where $V_{n,1}$ is the quantity that needs to be computed, and the other notation will be explained shortly. We should note that even $V_{n,1}$ was thought not to be easily calculated, however we will demonstrate that this can be done quite readily. We will also, in many cases, be able to give explicit expressions for

the more complex quantities. This also sets up some interesting relationships between various special functions, integral transforms, and probabilities.

We provide a discussion and definition of Fox H- and Meijer G-functions in the appendix, which is obtained from various sources. We now proceed to address some preliminaries and present a more specific outline.

1.1. Preliminaries

Again from Pitman (39), for Π_n an exchangeable partition of $\{1, 2, \ldots, n\}$ and a particular partition $\{A_1, \ldots, A_k\}$ of $\{1, 2, \ldots, n\}$ with $|A_i| = n_i$ for $1 \le i \le k$, where $n_i \ge 1$ and $\sum_{i=1}^k n_i = n$, then letting,

$$[a]_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},$$

the conditional EPPF, associated with the $\mathrm{PK}(\rho_\alpha|t)$ partition, is defined as $p_\alpha(n_1, \ldots, n_k|t) = \mathbb{P}(\Pi_n = \{A_1, \ldots, A_k\}|t)$, where

$$p_\alpha(n_1, \ldots, n_k|t) = \mathbb{G}_\alpha^{(n,k)}(t) \prod_{j=1}^{k} [1-\alpha]_{n_j-1} \tag{1.1}$$

and one can write

$$\begin{aligned} \mathbb{G}_\alpha^{(n,k)}(t) &= \frac{\alpha^k}{t^n \Gamma(n-k\alpha) f_\alpha(t)} \left[\int_0^t f_\alpha(t-v) v^{n-k\alpha-1} dv\right] \\ &= \frac{\alpha^k t^{-k\alpha}}{\Gamma(n-k\alpha) f_\alpha(t)} \left[\int_0^1 f_\alpha(tu)(1-u)^{n-k\alpha-1} du\right]. \end{aligned} \tag{1.2}$$

Using the terminology in (12), call the $\mathrm{PK}(\rho_\alpha|t)$ partitions the $(\alpha|t)$-partitions.

Now suppressing dependence on α and γ, for each n and k, set

$$V_{n,k} = \int_0^\infty \mathbb{G}_\alpha^{(n,k)}(t)\gamma(dt).$$

Pitman (39) shows that the EPPF of the $\mathrm{PK}(\rho_\alpha, \gamma)$ partition is given by

$$p_{\alpha,\gamma}(n_1, \ldots, n_k) = V_{n,k} \prod_{j=1}^{k} [1-\alpha]_{n_j-1}. \tag{1.3}$$

Note that by setting γ to be point mass at t, (1.3) equates with (1.1). The most well-known member of this class is the case where for $\theta > -\alpha$, γ corresponds to the distribution of the random variable $S_{\alpha,\theta}$ having density

$$f_{S_{\alpha,\theta}}(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)} t^{-\theta} f_\alpha(t) \tag{1.4}$$

and satisfies for $\delta+\theta>-\alpha$,

$$\mathbb{E}[S_{\alpha,\theta}^{-\delta}]=\frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)}\mathbb{E}[S_{\alpha}^{-(\delta+\theta)}]=\frac{\Gamma(\frac{(\theta+\delta)}{\alpha}+1)}{\Gamma(\theta+\delta+1)}\frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)}.$$

Note $S_{\alpha,0}\stackrel{d}{=}S_{\alpha}$. Furthermore, the random variables satisfy the remarkable identity that may be found in Pitman (40, section 4.2) and Perman, Pitman and Yor (35). That is, for any $\theta>-\alpha$,

$$\frac{1}{S_{\alpha,\theta}}\stackrel{d}{=}\frac{\beta_{\theta+\alpha,1-\alpha}}{S_{\alpha,\theta+\alpha}},\tag{1.5}$$

where $\beta_{\theta+\alpha,1-\alpha}$ is a beta$(\theta+\alpha,1-\alpha)$ random variable independent of $S_{\alpha,\theta}$. Note that the identity (1.5) plays an important role in the recent work of (18) and hence, through that work, will play a prominent role here. That is, using (1.4) as the mixing density gives the EPPF of the two parameter (α,θ) Poisson-Dirichlet distribution, say, $\mathrm{PD}(\alpha,\theta)$, given by

$$p_{\alpha,\theta}(n_1,\ldots,n_k)=\frac{\prod_{l=1}^{k}(\theta+l\alpha)}{[\theta+1]_{n-1}}\prod_{j=1}^{k}[1-\alpha]_{n_j-1}.\tag{1.6}$$

The quantity in (1.6) extends beyond the stable case, as it is also defined for the cases of $\alpha=0$ and $\theta>0$ corresponding to the famous result related to the Dirichlet process, otherwise known as the Chinese restaurant process, or Ewens $(0,\theta)$-partitions. The other possibility is the case of $-\infty\le\alpha<0$ and $\theta=m|\alpha|$, for $m=1,2\ldots$, referred to as $(\alpha,|\alpha|m)$ partitions. The $\mathrm{PD}(\alpha,\theta)$ plays an important role in a variety of diverse applications. See Pitman (40) for a general overview and set of references, and in particular, its relation to Bessel and Brownian phenomena. See Bertoin (2) for its role in terms of fragmentation and coagulation phenomena. See Ishwaran and James (15; 16) and Pitman (37) for some applications in Bayesian statistics.

A remarkable fact, see Gnedin and Pitman (12, Theorem 12) and Pitman (40, Theorem 4.6, p. 86), is that the EPPF's generated by (1.1), that is (1.3), and mixtures of the Ewens $(0,\theta)$-partitions and $(\alpha,|\alpha|m)$ partitions, constitute the only infinite EPPF's having Gibbs form, that is, infinite EPPF's of the form

$$c_{n,k}\prod_{j=1}^{k}w_{n_j}.$$

This, as discussed in regards to the $\mathrm{PD}(\alpha,\theta)$ family, has potential implications both from a practical and theoretical point of view in a variety of disciplines. In particular, the $(\alpha|t)$-partitions constitute the largest and most diverse of such classes. However, there are only a few examples where $V_{n,k}$ has been computed. Besides the $\mathrm{PD}(\alpha,\theta)$ case, there are also the models formed by taking γ as a density proportional to $\mathrm{e}^{-bt}f_{\alpha}(t)$. In addition, Pitman (39, section 8) and Pitman (40, section 4.5, p.90) show that conditional EPPF of the

$(1/2|t^{-2})$-partition, corresponding to the *Brownian excursion partition*, is such that $\mathbb{G}_{1/2}^{(n,k)}(t)$ can be expressed in terms of Hermite functions (25, section 10.2). This explicit result is due in part to the fact that $S_{1/2}$ is equivalent in distribution to an inverse gamma distribution with shape 1/2, and hence in contrast to the case of general S_α has a simple explicit density. However, clearly, given the fact that γ may be quite arbitrary, it is not enough to simply know the explicit form of the density of f_α.

1.2. Goals and Outline

Faced with this our goal becomes quite clear. Find methods to explicitly calculate and hopefully provide further interpretation of the quantities $\mathbb{G}_\alpha^{(n,k)}(t)$ and $V_{n,k}$ for general α. We wish to emphasize that we are not interested in merely suggesting crude numerical methods which do not have interpretability.

Our first task will be to provide an answer to a question posed by Pitman (40, Problem 4.3.3, p. 87), which goes beyond merely wanting a numerical calculation. We paraphrase it as follows,

> Pitman (40, section 4.5, eq. (4.59) and (4.67)) shows that in the case of $\alpha = 1/2$, the integral $\mathbb{G}_\alpha^{(n,k)}(t)$ can be simply expressed in terms of an entire function of a complex variable, the *Hermite function*, which has been extensively studied. It is natural to ask whether $\mathbb{G}_\alpha^{(n,k)}(t)$ might be similarly represented in terms of some entire functions with a parameter α, which reduces to the Hermite function for $\alpha = 1/2$.

In sections 2 and 3 of this paper we provide an answer to this question by showing that $\mathbb{G}_\alpha^{(n,k)}(t)$ can be expressed as the ratio of Fox H-functions in the case of general α and Meijer G-functions in the case where α is rational. In particular, in section 3.1, we recover the case of the Hermite function based on the calculus of Meijer G-functions when $\alpha = 1/2$, and show that for general rational α these may be expressed as ratios of sums of generalized hypergeometric functions. Additionally, in those 2 sections we show that $\mathbb{G}_\alpha^{(n,k)}(t)$ can be expressed in terms of densities derived from $S_{\alpha,k\alpha}$ and corresponding beta random variables. In section 4 we obtain results for the unconditional Gibbs models, that is, calculations for $V_{n,k}$. In section4.1 we show that one may use the calculus of Meijer G-functions to express many $V_{n,k}$ in terms of G-functions which are then readily computable. Sections 4.1.1 and 4.1.2 provide new specific examples of EPPF's. Section 5 represents our first real departure from calculations of EPPF's based on the theory of Meijer G-functions. In particular, we demonstrate that one can obtain calculations for all values of α using random variables connected to Lamperti (24), which have been recently studied in James (18). Interesting special examples, which can be computed by various other means, are given in sections 5.1 and 5.2. Section 6, albeit short, is a pivotal section as it demonstrates the important role of recursion formulae. This, as mentioned earlier, shows that one only need to focus on calculation of the $V_{n,1}$ terms in order to obtain the general $V_{n,k}$. As a side note, we believe that even in the case where $V_{n,k}$ may be represented in terms of G-functions and therefore computable, it

might be more efficient to use the recursion when one is interested in problems potentially involving the calculation of all $V_{n,k}$ for $k = 1, \ldots, n$ and $n = 1, 2, \ldots$. Such problems occur, for instance, in the implementation of generalized Chinese restaurant schemes as can be seen in Ishwaran and James (16) (see also Pitman (40, section 3.1). In those cases n represents sample size of data and can be in the thousands. In addition the recursion can be used to obtain new recursive relationships for various quantities, we will demonstrate that in section 8. In section 7, we describe purely probabilistic methods to calculate $V_{n,1}$ and in fact obtain expression for general $V_{n,k}$. We point out again that there are very few explicit examples of EPPF's so it is rather striking that we will now show how to obtain many of them. Section 7.1 introduces what we call the *Lamperti class* which can be seen as an extension of section 5. Section 7.2 introduces what we call the *Beta-Gamma class*. Remarkably this class shows that $V_{n,k}$ may be expressed in terms of expectations involving only beta and gamma random variables. Section 8 represents a non-trivial application of Proposition 7.4 which exploits the quite broad results in Pitman and Yor (44).

Remark 1.1. *Throughout we will write G_δ to represent a gamma$(\delta, 1)$ random variable and $\beta_{a,b}$ to represent a beta(a, b) random variable. Furthermore, unless otherwise stated, we will assume that when we write products of random variables, that means the individual random variables are independent.*

2. Conditional EPPF

From Schneider (46) (see also (27; 33)), one may represent the density of S_α in terms of an H-function as follows.

$$f_\alpha(t) = \frac{1}{\alpha} H^{1,1}_{2,2}\left[t \left| \begin{matrix} \left(1-\frac{1}{\alpha}, \frac{1}{\alpha}\right), (0,1) \\ \\ (0,1), (0,1) \end{matrix} \right. \right], \qquad t > 0. \tag{2.1}$$

Applying (9.7) in the appendix, one can write

$$f_\alpha(t) = \frac{1}{\alpha} H^{0,1}_{1,1}\left[t \left| \begin{matrix} (1-\frac{1}{\alpha}, \frac{1}{\alpha}) \\ \\ (0,1) \end{matrix} \right. \right], \qquad t > 0, \tag{2.2}$$

and use this to describe $\mathbb{G}^{(n,k)}_\alpha(t)$.

Theorem 2.1. *The function*

$$\mathbb{G}^{(n,k)}_\alpha(t)$$

defined by (1.2) appearing in (1.1) can be expressed as follows.

(i) $\mathbb{G}^{(n,k)}_\alpha(t)$ *is representable in terms of ratios of probability densities as,*

$$\mathbb{G}^{(n,k)}_\alpha(t) = \frac{\alpha^{k-1}\Gamma(k)}{\Gamma(n)} \frac{\tilde{f}_{\alpha,(n,k)}(t)}{f_\alpha(t)},$$

where $\tilde{f}_{\alpha,(n,k)}$ *denotes the density of the random variables*

$$\frac{S_{\alpha,k\alpha}}{\beta_{k\alpha,n-k\alpha}} \stackrel{d}{=} \frac{S_{\alpha,(k-1)\alpha}}{\beta_{(k-1)\alpha+1,n-1-(k-1)\alpha}}, \tag{2.3}$$

for $k = 1, \ldots, n$. *In particular, for* $k = 1$,

$$\frac{S_{\alpha}}{\beta_{1,n-1}} \stackrel{d}{=} \frac{S_{\alpha,\alpha}}{\beta_{\alpha,n-\alpha}}. \tag{2.4}$$

(ii) For all $0 < \alpha < 1$, $\mathbb{G}_{\alpha}^{(n,k)}(t)$ *is expressible as the ratio of Fox* H*-functions,*

$$\mathbb{G}_{\alpha}^{(n,k)}(t) = \alpha^{k} \frac{H_{1,1}^{0,1}\left[\, t \left|\begin{matrix}(1-\frac{1}{\alpha}-k, \frac{1}{\alpha}) \\ \\ (-n,1)\end{matrix}\right.\right]}{H_{1,1}^{0,1}\left[\, t \left|\begin{matrix}(1-\frac{1}{\alpha}, \frac{1}{\alpha}) \\ \\ (0,1)\end{matrix}\right.\right]}.$$

Proof. Let us proceed by first deriving the density of $S_{\alpha,k\alpha}/\beta_{k\alpha,n-k\alpha}$. First for clarity we use the fact that $\beta_{k\alpha,n-k\alpha}$ has density,

$$\frac{\Gamma(n)}{\Gamma(k\alpha)\Gamma(n-k\alpha)} u^{k\alpha-1}(1-u)^{n-k\alpha-1},$$

for $0 < u < 1$. Now setting $\theta = k\alpha$ in (1.4), we see that the usual operations to obtain the density involves

$$[u^{-k\alpha+1}t^{-k\alpha}] f_{\alpha}(tu) u^{k\alpha-1}(1-u)^{n-k\alpha-1}.$$

Hence due to the above cancelations, the density is

$$\tilde{f}_{\alpha,(n,k)}(t) = \frac{\Gamma(n)\alpha}{\Gamma(k)\Gamma(n-k\alpha)} t^{-k\alpha} \int_{0}^{1} f_{\alpha}(tu)(1-u)^{n-k\alpha-1} du.$$

Now it follows that,

$$\beta_{k\alpha,n-k\alpha} \stackrel{d}{=} \beta_{k\alpha+1-\alpha,n-k\alpha-1+\alpha}\beta_{k\alpha,1-\alpha}$$

Furthermore, setting $\theta = k\alpha$ in (1.5), one gets

$$\frac{S_{\alpha,k\alpha}}{\beta_{k\alpha,1-\alpha}} \stackrel{d}{=} S_{\alpha,k\alpha-\alpha}.$$

These two points yield the equivalence in (2.3) and (2.4).

In order to establish statement (ii), first write

$$\mathbb{H}_{\alpha}^{(n,k)}(t) = \frac{(tu)^{-k\alpha}}{\Gamma(n-k\alpha)} \int_{0}^{1} f_{\alpha}(tu) u^{k\alpha}(1-u)^{n-k\alpha-1} du.$$

Substituting the expression for $f_\alpha(ut)$ with (2.1), and then using (9.10) in the appendix to obtain an expression for $(tu)^{-k\alpha} f_\alpha(ut)$, one sees that

$$\begin{aligned}
\mathbb{H}_\alpha^{(n,k)}(t) &\equiv \frac{1}{\alpha\Gamma(n-k\alpha)} \int_0^1 u^{k\alpha}(1-u)^{n-k\alpha-1} \\
&\qquad\qquad \times H_{2,2}^{1,1}\left[tu \left| \begin{matrix} (1-\frac{1}{\alpha}-k, \frac{1}{\alpha}), (-k\alpha, 1) \\ \\ (-k\alpha, 1), (-k\alpha, 1) \end{matrix} \right. \right] du \\
&= \frac{1}{\alpha} H_{1,1}^{0,1}\left[t \left| \begin{matrix} (1-\frac{1}{\alpha}-k, \frac{1}{\alpha}) \\ \\ (-n, 1) \end{matrix} \right. \right], \qquad (2.5)
\end{aligned}$$

where the last equality follows from identity 2 in (45, p. 355) and some manipulations based on (9.11), (9.6) and (9.7). The result is concluded by applying the representation (2.2).

Statement (i) of Theorem 2.1 provides a probabilistic interpretation of $\mathbb{G}_\alpha^{(n,k)}(t)$. We shall see that, in particular, the distributional identity (2.4) will play a crucial role in applying probabilistic arguments to calculations based on $\mathbb{G}_\alpha^{(n,k)}(t)$. Statement (ii) expresses $\mathbb{G}_\alpha^{(n,k)}(t)$ in terms of Fox H-functions, which, among other things, allows one to make precise interpretations of it. Overall, Theorem 2.1 sets up a myriad of duality relationships between probabilistic quantities based on stable and beta random variables and a very large class of special functions. In particular, as we will show, one can use statement (i) to obtain explicit calculations for various Fox H-functions which are not yet readily computable by other means. Now when $\alpha = m/r$, the next result shows that expressions in statement (ii) reduces to ratios of Meijer G-functions. The significance being there is that calculations of Meijer G-functions are readily available in *Mathematica* and other mathematical softwares.

3. Rational Values, Products of Beta and Gamma Random variables and Meijer *G*-Functions

When $\alpha = m/r$ for integers $1 \le m < r$, the stable random variable $S_{m/r}$ can be represented in terms of independent beta and gamma random variables as follows. $\left(\frac{m}{S_{\frac{m}{r}}}\right)^m \stackrel{d}{=} r^r Z_{m,r}$, where

$$Z_{m,r} \stackrel{d}{=} \left(\prod_{i=1}^{m-1} \beta_{\frac{i}{r}, \frac{i}{m}-\frac{i}{r}}\right)\left(\prod_{j=m}^{r-1} G_{\frac{j}{r}}\right).$$

This result may be found in Chaumont and Yor (4, p.113) (see also (18, section 6)). Now, by Theorem 9 in Springer and Thompson (48, p.733), the density of

$Z_{m,r}$ can be represented in terms of Meijer G-functions as follows,

$$f_{Z_{m,r}}(t) = K_{m/r} G^{r-1,0}_{m-1,r-1}\left(t \,\middle|\, \begin{matrix} (\frac{i}{m}-1)_1^{m-1} \\ \\ (\frac{j}{r}-1)_1^{r-1} \end{matrix}\right), \tag{3.1}$$

where

$$K_{m/r} = \prod_{i=1}^{m-1} \frac{\Gamma\left(\frac{i}{m}\right)}{\Gamma\left(\frac{i}{r}\right)} \prod_{j=m}^{r-1} \frac{1}{\Gamma\left(\frac{j}{r}\right)}, \tag{3.2}$$

Hence the density of $S_{\frac{m}{r}}$ in s is given by

$$\begin{aligned} & f_{Z_{m,r}}\left(\frac{m^m}{r^r s^m}\right) \times \left|-\frac{m^{m+1}}{r^r s^{m+1}}\right| \\ &= K_{m/r} r^{\frac{r}{m}} G^{0,r-1}_{r-1,m-1}\left(\left(\frac{r^r}{m^m}\right) s^m \,\middle|\, \begin{matrix} (1-\frac{1}{m}-\frac{i}{r})_1^{r-1} \\ \\ (1-\frac{1}{m}-\frac{j}{m})_1^{m-1} \end{matrix}\right), \end{aligned} \tag{3.3}$$

by absorbing the term $(m^m r^{-r} s^{-m})^{(m+1)/m}$ using (9.10) and then applying (9.8). Note one could have used the result of (49) to obtain (3.3). However that result does not equate $S_{m/r}$ with beta and gamma random variables.

Now define the vectors,

$$\tau^{(k)}_{m,r} = \left(\left(1-\frac{1}{m}-\frac{i+k}{r}\right)_1^{r-1}, \left(\frac{i-1}{m}-\frac{k}{r}\right)_1^{m}\right) \tag{3.4}$$

and

$$\delta^{(k)}_{m,r} = \left(\left(1-\frac{1}{m}-\frac{j}{m}-\frac{k}{r}\right)_1^{m-1}, \left(\frac{j-1-n}{m}\right)_1^{m}\right). \tag{3.5}$$

Theorem 3.1. *Let m, r denote integers such that $1 \le m < r$. Then, for $\alpha = m/r$, (1.2) is expressible as*

$$\mathbb{G}^{(n,k)}_{\frac{m}{r}}(t) = \frac{m^{k-n} G^{0,m+r-1}_{m+r-1,2m-1}\left(\left(\frac{r^r}{m^m}\right) t^m \,\middle|\, \begin{matrix} \tau^{(k)}_{m,r} \\ \\ \delta^{(k)}_{m,r} \end{matrix}\right)}{G^{0,r-1}_{r-1,m-1}\left(\left(\frac{r^r}{m^m}\right) t^m \,\middle|\, \begin{matrix} (1-\frac{1}{m}-\frac{i}{r})_1^{r-1} \\ \\ (1-\frac{1}{m}-\frac{j}{m})_1^{m-1} \end{matrix}\right)},$$

or

$$\mathbb{G}^{(n,k)}_{\frac{m}{r}}\left(\left(\frac{m}{r^{\frac{r}{m}}}\right) s^{\frac{1}{m}}\right) = \frac{m^{k-n} G^{0,m+r-1}_{m+r-1,2m-1}\left(s \,\middle|\, \begin{matrix} \tau^{(k)}_{m,r} \\ \\ \delta^{(k)}_{m,r} \end{matrix}\right)}{G^{0,r-1}_{r-1,m-1}\left(s \,\middle|\, \begin{matrix} (1-\frac{1}{m}-\frac{i}{r})_1^{r-1} \\ \\ (1-\frac{1}{m}-\frac{j}{m})_1^{m-1} \end{matrix}\right)}.$$

Proof. From statement (i) in Theorem 2.1 it suffices to obtain the density of $S_{m/r,km/r} \times \beta_{km/r,n-km/r}$. Using the expression for $f_{m/r}$ in (3.3), we note that a solution for the integral $\int_0^1 f_{m/r}(tu)(1-u)^{q-1}du$, for $q>0$, is obtained from Corollary 9.1 as

$$\frac{K_{m/r} r^{\frac{r}{m}}\Gamma(q)}{m^q} G^{0,m+r-1}_{m+r-1,2m-1}\left(\left(\frac{r^r}{m^m}\right)t^m \,\middle|\, \begin{matrix} (1-\frac{1}{m}-\frac{i}{r})_1^{r-1}, (\frac{i-1}{m})_1^m \\ (1-\frac{1}{m}-\frac{j}{m})_1^{m-1}, (\frac{j-1-q}{m})_1^m \end{matrix}\right).$$

Hence, absorbing the term $t^{-km/r}$ into the above expression with $q=n-km/r$ by (9.10) and taking the ratio of the resulting expression and (3.3) gives $\mathbb{G}^{(n,k)}_{m/r}(t)$. The second result follows from simple algebra.

We next show how to recover the result of Pitman (40, Corollary 4.11, p. 93) or (39) in the case of $m=1$ and $r=2$, and show that the other rational cases of α may be expressed in terms of generalized hypergeometric functions.

3.1. Hermite and generalized hypergeometric functions

(i) When $m=1$ and $r=2$, we see that Theorem 3.1 recovers almost immediately the result of Pitman (40, Corollary 4.11, p. 93) or (39) as follows. One has that $\mathbb{G}^{(n,k)}_{1/2}(t)$ is expressible as

$$\frac{G^{0,2}_{2,1}\left(4t \,\middle|\, \begin{matrix} -\frac{1+k}{2}, -\frac{k}{2} \\ -n \end{matrix}\right)}{G^{0,1}_{1,0}\left(4t \,\middle|\, \begin{matrix} -\frac{1}{2} \\ \text{---} \end{matrix}\right)} = \frac{(4t)^{-\frac{1+k}{2}-1} e^{-\frac{1}{4t}} U\left(-\frac{k}{2}+n, \frac{3}{2}, \frac{1}{4t}\right)}{(4t)^{-\frac{3}{2}} e^{-\frac{1}{4t}}},$$

where $U(a,b,c)$ is the confluent hypergeometric function of the second kind (see (25, p. 263)). The above ratio reduces to

$$2^{-k+1} t^{-\frac{k}{2}+\frac{1}{2}} U\left(-\frac{k}{2}-\frac{1}{2}+n, \frac{1}{2}, \frac{1}{4t}\right)$$

via an application of the recurrence relation (47, p. 505)

$$U(a,b,z) = z^{1-b} U(1+a-b, 2-b, z).$$

A change of variable $t=\frac{1}{2}\lambda^{-2}$ yields the expression $2^{n-k}\lambda^{k-1}h_{k+1-2n}(\lambda)$ inside equation (110) in (39), where $h_\nu(\lambda)$ is the Hermite function of index ν (25, section 10.2), based on the following relationship,

$$h_\nu(\lambda) = 2^{\nu/2} U\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{\lambda^2}{2}\right).$$

(ii) Now, when $m = 1$ and $r = 3$, $\mathbb{G}_{1/3}^{(n,k)}(t)$ is expressible as

$$\frac{G_{3,1}^{0,3}\left(27t \middle| \begin{matrix} -\frac{2+k}{3}, -\frac{1+k}{3}, -\frac{k}{3} \\ -n \end{matrix}\right)}{G_{2,0}^{0,2}\left(27t \middle| \begin{matrix} -\frac{1}{3}, -\frac{2}{3} \\ \text{---} \end{matrix}\right)},$$

where the G-functions at the numerator and at the denominator are respectively

$$\frac{4\pi^2}{3^{\frac{k}{3}+4}t^{\frac{k}{3}+1}}\left[\frac{{}_1F_2(1-n+\frac{k}{3};\frac{1}{3},\frac{2}{3};\frac{1}{27t})}{\Gamma(n-\frac{k}{3})\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} - \frac{{}_1F_2(1-n+\frac{k}{3}+\frac{1}{3};\frac{2}{3},\frac{4}{3};\frac{1}{27t})}{3t^{\frac{1}{3}}\Gamma(n-\frac{k+1}{3})\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})} + \frac{{}_1F_2(1-n+\frac{k}{3}+\frac{2}{3};\frac{4}{3},\frac{5}{3};\frac{1}{27t})}{(3t^{\frac{1}{3}})^2\Gamma(n-\frac{k+2}{3})\Gamma(\frac{4}{3})\Gamma(\frac{5}{3})}\right]$$

and

$$\frac{2\pi}{3^{\frac{9}{2}}t^{\frac{4}{3}}}\left[\frac{{}_0F_1(;\frac{2}{3};\frac{1}{27t})}{\Gamma(\frac{2}{3})} - \frac{{}_0F_1(;\frac{4}{3};\frac{1}{27t})}{3t^{\frac{1}{3}}\Gamma(\frac{4}{3})}\right],$$

wherein, for non-negative integers p and q, $p \le q$ or $p = q+1$, $|z| \le 1$, and $b_j \neq 0, -1, -2, \ldots j = 1, \ldots, q$,

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{i=0}^{\infty} \frac{[a_1]_i [a_2]_i \cdots [a_p]_i}{[b_1]_i [b_2]_i \cdots [b_q]_i} \frac{z^i}{i!}, \tag{3.6}$$

is called the generalized hypergeometric function (see (6, Chapter IV), (45, p. 437)), which is available in *Mathematica* as `HypergeometricPFQ[{a1,...,ap},{b1,...,bq},z]`.

Remark 3.1. *In general, when $\alpha = m/r$ with $r > 2$, both the numerator and the denominator of the ratios in Theorem 3.1 are expressible in terms of sums of generalized hypergeometric functions of the form,*

$${}_{2m-1}F_{m+r-2}(a_1, \ldots, a_{2m-1}; b_1, \ldots, b_{m+r-2}; \cdot)$$

and

$${}_{m-1}F_{r-2}(a_1, \ldots, a_{m-1}; b_1, \ldots, b_{r-2}; \cdot)$$

respectively.

4. Unconditional Gibbs

Hereafter we assume that the mixing distribution can be represented as the density

$$\gamma_\alpha(t) = h(t) f_\alpha(t)$$

where $h(t)$ is a non-negative function such that

$$\int_0^\infty h(t) f_\alpha(t) = \mathbb{E}[h(S_\alpha)] = 1.$$

Note that mixing the density γ_α over $\mathbb{G}_\alpha^{(n,k)}(t)$ leads to the following class of operators

$$\mathcal{I}_\alpha^{(n,k)}(h) = \int_0^\infty \mathbb{G}_\alpha^{(n,k)}(t)\gamma_\alpha(t)dt = \alpha^{k-1}\int_0^\infty h(t) H_{1,1}^{0,1}\left[t \,\middle|\, \begin{matrix} (1-\frac{1}{\alpha}-k, \frac{1}{\alpha}) \\ \\ (-n,1) \end{matrix} \right] dt$$

which is a particular type of Fox H-integral transforms, see, for instance, (22) and (20).

Theorem 4.1. *Suppose that $h(t)$ is a non-negative integrable function with respect to f_α. Then, without loss of generality, set $\int_0^\infty h(t) f_\alpha(t) = 1$, and form the density $\gamma_\alpha(t) = h(t) f_\alpha(t)$. Then the EPPF of $\mathrm{PK}(\rho_\alpha; \gamma_\alpha)$ random partition has Gibbs form*

$$p_{\alpha,\gamma_\alpha}(n_1, \ldots, n_k) = V_{n,k} \prod_{j=1}^k [1-\alpha]_{n_j - 1}$$

where, for $k = 1, \ldots, n$,

$$\begin{aligned} V_{n,k} &= \frac{\alpha^{k-1}\Gamma(k)}{\Gamma(n)} \mathbb{E}\left[h\left(\frac{S_{\alpha,k\alpha}}{\beta_{k\alpha,n-k\alpha}} \right) \right] \\ &= \frac{\alpha^{k-1}\Gamma(k)}{\Gamma(n)} \mathbb{E}\left[h\left(\frac{S_{\alpha,(k-1)\alpha}}{\beta_{(k-1)\alpha+1,n-1-(k-1)\alpha}} \right) \right] \\ &= \mathcal{I}_\alpha^{(n,k)}(h) \end{aligned}$$

with $V_{1,1} = 1$.

This result follows directly from Theorem 2.1.

4.1. G-*transform for rational values*

When $h(t)$ is set to be a G-function, one can use, for instance, Theorem 9.1 in the appendix, to calculate many EPPF's as follows.

Proposition 4.1. *When $\alpha = m/r$ and $h(t)$ is expressible as*

$$C \times G_{w,x}^{u,v}\left(\sigma t \,\middle|\, \begin{matrix} (c_i)_1^w \\ \\ (d_j)_1^x \end{matrix} \right),$$

where C is a constant and notation in (9.17) follows, then $V_{n,k}$ is

$$\frac{CK_{m/r}r^{r/m}m^{\rho+(x-w)-1+k-n}}{\sigma(2\pi)^{b^*(m-1)}} \times G^{vm,(u+1)m+r-1}_{(x+1)m+r-1,(w+2)m-1}\left(\frac{r^r}{(\sigma m)^m}\middle|\begin{matrix}\tau^{(k)}_{m,r},\Delta(m,-d_1),\ldots,\Delta(m,-d_x)\\ \delta^{(k)}_{m,r},\Delta(m,-c_1),\ldots,\Delta(m,-c_w)\end{matrix}\right),$$

where $K_{m/r}$ is defined in (3.2), $\Delta(\ell,a)$, for any integer ℓ, is defined in (9.19), and ρ and b^* are defined for $G^{u,v}_{w,x}(\sigma t)$ according to (9.17).

The result is just a specialization of Theorem 9.1 given in the appendix. We now address two specific new examples.

4.1.1. Example: Modified Bessel functions

Proposition 4.2. *Suppose we take $\gamma_{1/2}(t)\propto K_\eta(\sqrt{t})f_{1/2}(t)$, or in other words, $h(t)=K_\eta(\sqrt{t})/\int K_\eta(\sqrt{t})f_{1/2}(t)dt$, where $K_\eta(\cdot)$ is the modified Bessel function of the third kind or Macdonald function (7, Sec. 7.2.1). Then,*

$$V_{n,k}=\frac{G^{0,4}_{4,1}\left(16\middle|\begin{matrix}-\frac{1+k}{2},-\frac{k}{2},\frac{\eta}{2},-\frac{\eta}{2}\\ -n\end{matrix}\right)}{G^{0,3}_{3,0}\left(16\middle|\begin{matrix}-\frac{1}{2},\frac{\eta}{2},-\frac{\eta}{2}\\ \rule{2em}{0.4pt}\end{matrix}\right)}.$$

Proof. The result follows from Proposition 4.1 by substituting $m=1$ and $r=1$ and recognizing

$$h(t)=C\times K_\eta(\sqrt{t})=C\times G^{2,0}_{0,2}\left(\frac{t}{4}\middle|\begin{matrix}\rule{2em}{0.4pt}\\ -\frac{\eta}{2},\frac{\eta}{2}\end{matrix}\right)$$

by (9.15), where

$$\begin{aligned}C^{-1} &= \int K_\eta(\sqrt{t})f_{1/2}(t)dt\\ &= \frac{4}{\sqrt{\pi}}\int_0^\infty G^{2,0}_{0,2}\left(\frac{t}{4}\middle|\begin{matrix}\rule{2em}{0.4pt}\\ -\frac{\eta}{2},\frac{\eta}{2}\end{matrix}\right)G^{0,1}_{1,0}\left(4t\middle|\begin{matrix}-\frac{1}{2}\\ \rule{2em}{0.4pt}\end{matrix}\right)dt\\ &= \frac{16}{\sqrt{\pi}}G^{0,3}_{3,0}\left(16\middle|\begin{matrix}-\frac{1}{2},\frac{\eta}{2},-\frac{\eta}{2}\\ \rule{2em}{0.4pt}\end{matrix}\right)\end{aligned}$$

due to Theorem 9.1.

4.1.2. Generalized Hypergeometric Functions as EPPF

An EPPF in terms of one generalized hypergeometric function defined in (3.6) (see (6, Chapter IV)) results when $V_{n,k} = \int_0^\infty \mathbb{G}_\alpha^{(n,k)}(t)\gamma(t)dt$ is representable as some constant multiplies either

$$G^{1,p}_{p,q+1}(-z), \quad \text{for } p \leq q. \tag{4.1}$$

or

$$G^{1,q+1}_{q+1,q+1}(-z), \quad \text{for } |z| \leq 1, \tag{4.2}$$

due to (9.16).

Proposition 4.3. *Suppose that $\alpha = 1/r$ and $h(t) = C \times G^{x,1}_{w,x}\left((r^r)\, t \left| \begin{matrix} (c_i)_1^w \\ \\ (d_j)_1^x \end{matrix} \right. \right)$, where C is a constant, notation in (9.17) follows, and none of $(c_i)_1^w$ is identical to any of $(d_j)_1^x$. When*

(i) $w \geq x + r$, or
(ii) $x = w - r + 1$ and $w \geq r - 1$,

$V_{n,k}$ is given by

$$C^*_{w,x} \times {}_{x+r}F_w \left(\left(\frac{i+k}{r} + \nu \right)_1^{r-1}, \frac{k}{r} + \nu, (d_i + \nu)_1^x ; (c_j + \nu)_1^w ; -1 \right),$$

where $\nu = 1 - n$ and

$$C^*_{w,x} = \frac{C K_{1/r} r^r \prod_{i=1}^{r-1} \Gamma\left(\frac{i+k}{r} + \nu\right) \Gamma\left(\frac{k}{r} + \nu\right) \prod_{i=1}^{x} \Gamma(d_i + \nu)}{\prod_{j=1}^{w} \Gamma(c_j + \nu)},$$

with $K_{1/r}$ defined in (3.2) with $m = 1$, provided that $n \neq d_j \neq \frac{k+\ell-1}{r}$ and $c_i \neq \frac{k+\ell-1}{r}$ for $i = 1, \ldots, w, j = 1, \ldots, x, \ell = 1, \ldots, r$.

Proof. It follows from Proposition 4.1 with $m = 1$ that $V_{n,k}$ is equal to $C K_{1/r} r^r$ multiplies

$$\begin{aligned} & G^{1,x+r}_{x+r,w+1}\left(1 \left| \begin{matrix} (-\frac{i+k}{r})_1^{r-1}, -\frac{k}{r}, (-d_i)_1^x \\ \\ -n, (-c_j)_1^w \end{matrix} \right. \right) \\ & \quad = G^{1,x+r}_{x+r,w+1}\left(1 \left| \begin{matrix} \left(1 - \left(\frac{i+k}{r} + \nu\right)\right)_1^{r-1}, 1 - \left(\frac{k}{r} + \nu\right), (1 - (d_i + \nu))_1^x \\ \\ 0, (1 - (c_j + \nu))_1^w \end{matrix} \right. \right), \end{aligned}$$

followed from (9.10). The last G-function takes either the form of (4.1) when $w \geq x + r$, or the form of (4.2) when $x = w - r + 1$ and $w \geq r - 1$, and, hence, the result follows from (9.16).

5. First distribution theory example: $S_{\alpha,\theta}$ given $X_{\alpha,\theta}$

We now come to our first result which does not rely on special properties of G- or H-functions and importantly applies to all values of α. Let S_α and $S_{\alpha,\theta}$ denote independent random variables having laws described previously. Then, define the random variables

$$X_{\alpha,\theta} = \frac{S_\alpha}{S_{\alpha,\theta}} \tag{5.1}$$

whose laws have been recently studied in James (18). They represent a natural generalization of the random variable

$$X_\alpha = \frac{S_\alpha}{S'_\alpha}$$

where $X_\alpha \stackrel{d}{=} X_{\alpha,0}$ and S'_α is independent of S_α and has the same distribution. Remarkably although S_α does not have a simple density, except for $\alpha = 1/2$, Lamperti (24) (see also Chaumont and Yor (4, exercise 4.2.1)) shows that the density of X_α is

$$f_{X_\alpha}(y) = \frac{\sin(\pi\alpha)}{\pi} \frac{y^{\alpha-1}}{y^{2\alpha} + 2y^\alpha \cos(\pi\alpha) + 1}, \qquad \text{for } y > 0, \tag{5.2}$$

with cdf

$$F_{X_\alpha}(y) = 1 - \frac{1}{\pi\alpha} \cot^{-1}\left(\cot(\pi\alpha) + \frac{y^\alpha}{\sin(\pi\alpha)}\right).$$

Note furthermore, when $\alpha = 1/2$,

$$X_{1/2} \stackrel{d}{=} \frac{G'_{1/2}}{G_{1/2}} \text{ and } X_{1/2,\theta} \stackrel{d}{=} \frac{G_{\theta+1/2}}{G_{1/2}}, \tag{5.3}$$

where $G'_{1/2} \stackrel{d}{=} G_{1/2}$ and all the gamma random variables are independent. See (17, section 4.2) for more on the variables (5.3).

Here we investigate the mixing distribution γ corresponding to the law of $S_{\alpha,\theta}$ given $X_{\alpha,\theta} = 1$. That is, the random variable with density

$$\gamma_\alpha(t) = C_{\alpha,\theta} t^{1-\theta} f_\alpha(t) f_\alpha(t) \tag{5.4}$$

where

$$C_{\alpha,\theta} = 1/f_{X_{\alpha,\theta}}(1).$$

So from (5.2),

$$C_{\alpha,0} = \frac{2\pi(1 + \cos(\pi\alpha))}{\sin(\pi\alpha)}.$$

Now define, for $\theta > 0$,

$$\Delta_\theta(x|F_{X_\alpha}) = \frac{1}{\pi} \frac{\sin(\pi\theta F_{X_\alpha}(x))}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^{\frac{\theta}{2\alpha}}}.$$

We now give a brief description of the density of $X_{\alpha,\theta}$ for $\theta > 0$ and general α which is due to James (18, Theorem 3.1). When $\theta = 1$,

$$f_{X_{\alpha,1}}(y) = \Delta_1(y|F_{X_\alpha}) = \frac{1}{\pi}\frac{\sin(\pi F_{X_\alpha}(y))}{[y^{2\alpha} + 2y^\alpha \cos(\alpha\pi) + 1]^{\frac{1}{2\alpha}}}.$$

Hence, the normalizing constant in this case satisfies

$$1/C_{\alpha,1} = f_{X_{\alpha,1}}(1) = \frac{1}{\pi}\frac{\sin(\pi F_{X_\alpha}(1))}{[2(1 + \cos(\alpha\pi))]^{\frac{1}{2\alpha}}}$$

In general, for $\theta > 0$,

$$f_{X_{\alpha,\theta}}(y) = \int_0^y (y - x)^{\theta-1} \Delta'_\theta(x)dx \tag{5.5}$$

where, suppressing dependence on F_{X_α}, Δ' denotes the derivative of Δ. Furthermore, define

$$\vartheta_j^{(n,k)}(x) = \frac{[\sin(\pi\alpha)]^{k-j}}{\pi}\frac{(x^\alpha \cos(\alpha\pi) + 1)^j x^{\alpha(k-j)}}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^k}. \tag{5.6}$$

Before we proceed with the description of the EPPF, we provide a representation of $\Delta_{k\alpha}$ connected with the random variable $X_{\alpha,k\alpha}$.

Lemma 5.1. *For $0 < \alpha < 1$, and $k = 1, 2, \ldots,$*

$$\Delta_{k\alpha}(x|F_{X_\alpha}) = \sum_{j=0}^{k} \binom{k}{j} \sin\left(\frac{\pi}{2}(k - j)\right) \vartheta_j^{(n,k)}(x)$$

Proof. Since $k = 1, 2, \ldots,$ we first apply the multiple angle formula to

$$\sin(\pi k\alpha F_{X_\alpha}(x)).$$

The result is concluded by noting the following identities which are given in James (18, Proposition 2.1),

$$\sin(\pi\alpha F_{X_\alpha}(x)) = \frac{x^\alpha \sin(\alpha\pi)}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^{1/2}}$$

and

$$\cos(\pi\alpha F_{X_\alpha}(x)) = \frac{x^\alpha \cos(\alpha\pi) + 1}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^{1/2}}.$$

Then we have the general description of the EPPF.

Proposition 5.1. *Suppose γ_α is specified as (5.4) for $\theta > -\alpha$. Then, for $n = 2, 3 \ldots,$ and $k = 1, \ldots, n$,*

$$V_{n,k} = C_{\alpha,\theta}\frac{\alpha^{k-1}\Gamma(\theta/\alpha + k)}{\Gamma(\theta + n - 1)} \int_0^1 (1 - x)^{n+\theta-2} \Delta_{\theta+k\alpha}(x|F_{X_\alpha})dx$$

(i) In particular, when $\theta = 0$, we obtain

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(k)}{\Gamma(n-1)} \sum_{j=0}^{k} \binom{k}{j} \sin\left(\frac{\pi}{2}(k-j)\right) \varphi_j^{(n,k)} \tag{5.7}$$

where

$$\varphi_j^{(n,k)} = \frac{2\pi(1+\cos(\alpha\pi))}{\sin(\pi\alpha)} \int_0^1 (1-x)^{n-2} \vartheta_j^{(n,k)}(x)dx.$$

Note that $\vartheta_j^{(n,k)}(x) > 0$ for $0 < x < 1$.

(ii) When $\theta = 0$ and $n > 1$,

$$V_{n,1} = \frac{2(1+\cos(\alpha\pi))}{\Gamma(n-1)} \int_0^1 \frac{(1-x)^{n-2} x^{\alpha}}{[x^{2\alpha} + 2x^{\alpha}\cos(\alpha\pi) + 1]} dx. \tag{5.8}$$

Proof. It is easy enough to work directly with the expression

$$\mathbb{G}_{\alpha}^{(n,k)}(t) = \frac{\alpha^k t^{-k\alpha}}{\Gamma(n-k\alpha) f_{\alpha}(t)} \left[\int_0^1 f_{\alpha}(tu)(1-u)^{n-k\alpha-1} du\right].$$

Mixing relative to $\gamma_{\alpha}(t)$ defined in (5.4), one sees that

$$\int_0^{\infty} t^{1-(\theta+k\alpha)} f_{\alpha}(tu) f_{\alpha}(t) dt = \frac{\Gamma(\theta/\alpha + k + 1)}{\Gamma(\theta + k\alpha + 1)} f_{X_{\alpha,\theta+k\alpha}}(u).$$

Now noting the form of the density of $X_{\alpha,\theta+k\alpha}$ from (5.5) and integrating with respect to u lead to the evaluation of the integral

$$\int_x^1 (u-x)^{\theta+k\alpha-1}(1-u)^{n-k\alpha-1} du = (1-x)^{n+\theta-1} \frac{\Gamma(\theta+k\alpha)\Gamma(n-k\alpha)}{\Gamma(\theta+n)}.$$

Ignoring constants for a moment this leads to an integral of the form

$$\int_0^1 (1-x)^{n+\theta-1} \Delta'_{\theta+k\alpha}(x) dx$$

Now, since $\theta > -\alpha$, it follows that $n + \theta > 1$ when $n = 2, 3, \ldots$. This allows us to use integration by parts to get

$$\int_0^1 (1-x)^{n+\theta-1} \Delta'_{\theta+k\alpha}(x) dx = (n+\theta-1) \int_0^1 (1-x)^{n+\theta-2} \Delta_{\theta+k\alpha}(x|F_{X_{\alpha}}) dx$$

which yields the general expression for $\theta > -\alpha$. Statements (i) and (ii) then follow from an application of Lemma 5.1, and also the use of (5.2).

When $\alpha = m/r$, we may easily express $V_{n,k}$ in terms of G-functions, which we leave to the reader. We focus on two interesting cases.

5.1. ${}_2F_1$ EPPF

When $\alpha = 1/2$, we have that for $\theta > -1/2$,

$$C_{1/2,\theta} = \frac{2^{1-\theta}\pi}{\Gamma(\theta+1)}.$$

One can then show that

$$V_{n,k} = \frac{2^{\theta+1}}{\Gamma(\theta+1)} G^{1,2}_{2,2}\left(1 \middle| \begin{array}{c} -\theta - \frac{k}{2}, -\theta - \frac{k}{2} + \frac{1}{2} \\ \\ 0, -\theta - n + \frac{1}{2} \end{array}\right).$$

Now applying (9.16) or identity (2.9.15) in (20), the last G-function reduces to

$$\frac{\Gamma(\theta + \frac{k}{2} + 1)\Gamma(\theta + \frac{k}{2} + \frac{1}{2})}{\Gamma(\theta + n + \frac{1}{2})} {}_2F_1\left(\theta + \frac{k}{2} + 1, \theta + \frac{k}{2} + \frac{1}{2}; \theta + n + \frac{1}{2}; -1\right),$$

where

$${}_2F_1\left(a, b; c; x\right) = \sum_{i=0}^{\infty} \frac{[a]_i [b]_i}{[c]_i} \frac{z^i}{i!} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{r^{b-1}(1-r)^{c-b-1}}{(1-xr)^a} dr$$

is the Gauss hypergeometric function (see (34)), which is a special case of (3.6). Note this result can also be checked by using the explicit density of $X_{1/2,\theta}$.

5.2. ${}_3F_2$ EPPF

When $\alpha = 1/3$, that is, $m = 1$ and $r = 3$, one gets

$$V_{n,k} = \frac{C_{1/3,\theta} 3^{3\theta}}{[\Gamma(1/3)\Gamma(2/3)]^2} G^{2,3}_{3,3}\left(1 \middle| \begin{array}{c} -\frac{k}{3} - \frac{2}{3}, -\frac{k}{3} - \frac{1}{3}, -\frac{k}{3} \\ \\ \theta - \frac{2}{3}, \theta - \frac{1}{3}, -n \end{array}\right).$$

Furthermore, the G-function reduces to

$$\frac{\Gamma(\theta + \frac{k}{3} + 1)\prod_{i=1}^{4}\Gamma(\theta + \frac{k}{3} + \frac{i}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(n + \theta + \frac{1}{3})\Gamma(2(\theta + \frac{k+3}{3}))} {}_3F_2\left(\begin{array}{c} \theta + \frac{k+2}{3}, n - \frac{k}{3}, \theta + \frac{k+3}{3} \\ \\ n + \theta + \frac{1}{3}, 2(\theta + \frac{k+3}{3}) \end{array}; 1\right).$$

6. A recursive method for calculating EPPF's and Fox H-transforms

Definition 3 or equation (8) of Gnedin and Pitman (12) establishes the following backward recursion for all $V_{n,k}$, $n = 1, 2, \ldots; k = 1, 2, \ldots, n$,

$$V_{n,k} = (n - k\alpha)V_{n+1,k} + V_{n+1,k+1} \tag{6.1}$$

with $V_{1,1} = 1$.

One can turn (6.1) around to obtain the following forward recursion,

$$V_{n+1,k+1} = V_{n,k} - (n - k\alpha)V_{n+1,k}. \tag{6.2}$$

The key point about the recursion (6.2) is that it enables computations of all $V_{n,k}$ for $n = 1, 2, \ldots; k = 1, 2, \ldots, n$, provided that

$$V_{n,1} = \frac{1}{\Gamma(n)} \mathbb{E}\left[h\left(\frac{S_\alpha}{\beta_{1,n-1}}\right)\right],$$

for $n = 1, 2, \ldots$, are known. For clarity, we demonstrate this point by considering the case of $n = 3$. When $V_{1,1}, V_{2,1}$ and $V_{3,1}$ are given, one can obtain all $V_{n,k}$ for $n = 1, 2, 3, k = 1, \ldots, n$ as follows.

(i) compute $V_{2,2} = V_{1,1} - (1-\alpha)V_{2,1}$;
(ii) compute $V_{3,2} = V_{2,1} - (2-\alpha)V_{2,2}$; and
(iii) compute $V_{3,3} = V_{2,2} - (2-2\alpha)V_{3,2}$.

So quite fortunately we can focus on the relatively simpler task of calculating $V_{n,1}$. The simplicity occurs because the quantities only depend on the distribution of the independent pairs $(S_\alpha, \beta_{1,n-1})$, or equivalently, $(S_{\alpha,\alpha}, \beta_{\alpha,n-\alpha})$. In particular, when S_α does not depend on k or n, we show some interesting applications of the recursion in section 8. One may also see how this applies to the explicit expressions in (5.8) and (5.7).

6.1. An all purpose solution?

It is widely believed that beyond the infinite series representation of f_α, there are no general explicit representations of f_α. In fact this is false as one may use the representation of Kanter (19), which we now describe. Setting

$$\mathrm{K}_\alpha(u) = \left[\frac{\sin(\pi\alpha u)}{\sin(\pi u)}\right]^{-\frac{1}{1-\alpha}} \left[\frac{\sin((1-\alpha)\pi u)}{\sin(\pi\alpha u)}\right],$$

it follows from Kanter (19) (see also Devroye (5) and Zolotarev (50)) that

$$f_\alpha(s) = \frac{\alpha}{1-\alpha} s^{-1/(1-\alpha)} \int_0^1 \mathrm{e}^{-s^{-\frac{\alpha}{1-\alpha}}\mathrm{K}_\alpha(u)} \mathrm{K}_\alpha(u) du. \tag{6.3}$$

That is,

$$S_\alpha \overset{d}{=} (\mathrm{K}_\alpha(U)/G_1)^{(1-\alpha)/\alpha}, \tag{6.4}$$

where U is a uniform random variable on $[0, 1]$, independent of a standard exponential variable G_1. So, using (6.4), one can obtain calculations for $V_{n,1}$ and hence $V_{n,k}$ by simulating U, G_1 and a $\beta_{1,n-1}$ variable. In addition, one can certainly obtain new integral representations for $V_{n,1}$ and $V_{n,k}$.

However, while this is certainly true in principle, and useful in some cases, there is still quite a bit lacking in this representation from an analytic point

of view. For instance, it is not obvious how to use (6.3) to obtain the Laplace transform of S_α or to obtain the nice form of the density of X_α. In addition, one would require that $h(t)$ has a manageable form, which is not the case for instance in section 8. Recall that even the known tractable case of $\alpha = 1/2$ does not always immediately lead to nice expressions for $V_{n,1}$. We will see in the next sections that one can do quite nicely without resorting to (6.3).

7. Classes of EPPF's via Probability Transforms

The first sections focused on the fact that we could use the fact that Mejier G-transforms can now be easily computed to calculate $V_{n,k}$ and related quantities. We also noted that while in theory the $V_{n,k}$ can be calculated based on H-transforms, the practical tools to do this are not yet available as not all H-functions can be computed easily without resorting to numerical integrations. We now demonstrate through using probability distribution theory how to calculate $V_{n,1}$ and hence by the recursion all $(V_{n,k})$. This in turn will provide new methods to calculate many H-transforms as well as G-transforms. We will focus on three general classes, but certainly more can be constructed.

7.1. Lamperti Class

Here, we introduce an entire class of EPPF's based on X_α and $X_{\alpha,k\alpha}$. First, for a positive integrable function g, let

$$1/L_\alpha = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{g(y) y^{\alpha-1} dy}{y^{2\alpha} + 2y^\alpha \cos(\pi\alpha) + 1} = \mathbb{E}[g(X_\alpha)].$$

Proposition 7.1. *Suppose that for each* $0 < \alpha < 1$, $h(t) = L_\alpha \mathbb{E}[g(S'_\alpha/t)]$. *That is,*

$$\gamma_\alpha(t) = L_\alpha \mathbb{E}[g(S'_\alpha/t)] f_\alpha(t).$$

Then, $\text{PK}(\rho_\alpha, \gamma_\alpha)$ *has the following properties.*

(i) $V_{n,1} = \dfrac{1}{\Gamma(n)} L_\alpha \mathbb{E}[g(X_\alpha \beta_{1,n-1})]$.

(ii) For $k = 1, 2, \ldots, n$,

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(k)}{\Gamma(n)} L_\alpha \mathbb{E}[g(X_{\alpha,k\alpha}\beta_{k\alpha,n-k\alpha})].$$

(iii) Equivalently,

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(k)}{\Gamma(n)} L_\alpha \mathbb{E}\left[g\left(\beta_{k,n-k} \sum_{j=1}^{k} X_\alpha^{(j)} D_j \right) \right]$$

where $(X_\alpha^{(j)})$ *are iid random variables equal in distribution to* X_α *and, independently,* $(D_1, \ldots D_k)$ *is a* k*-variate Dirichlet*$(1, \ldots, 1)$ *random vector.*

Proof. Statements (i) and (ii) follow from Theorem 2.1 and the definition,

$$X_{\alpha,k\alpha} = \frac{S'_\alpha}{S_{\alpha,k\alpha}}.$$

Statement (iii) amounts to showing that

$$X_{\alpha,k\alpha}\beta_{k\alpha,n-k\alpha} \stackrel{d}{=} \beta_{k,n-k}\sum_{j=1}^{k} X_\alpha^{(j)} D_j.$$

In order to do this, we refer back to the recent work of James (18; 17). First, write

$$\beta_{k\alpha,n-k\alpha} \stackrel{d}{=} \beta_{k,n-k}\beta_{k\alpha,k(1-\alpha)}.$$

Then using the notation and the result in (18, Proposition 3.1(ii)), $X_{\alpha,k\alpha} \stackrel{d}{=} M_{k\alpha}(F_{X_\alpha})$. That is, $X_{\alpha,k\alpha}$ is a Dirichlet mean random variable of order $k\alpha$ indexed by F_{X_α}. Let Y_α denote a Bernoulli random variable with success probability α, independent of X_α. Now, applying Theorem 2.1 in James (17), it follows that

$$\beta_{k\alpha,k(1-\alpha)}M_{k\alpha}(F_{X_\alpha}) \stackrel{d}{=} M_k(F_{X_\alpha Y_\alpha})$$

where $M_k(F_{X_\alpha Y_\alpha})$ denotes a Dirichlet mean of order k indexed by the cdf of $X_\alpha Y_\alpha$, $F_{X_\alpha Y_\alpha}$. Again, these specific random variables are found in (18). Now an application of Proposition 9 in Hjort and Ongaro (14) (see also Proposition 4.5 in (18)) yields

$$M_k(F_{X_\alpha Y_\alpha}) \stackrel{d}{=} \sum_{j=1}^{k} D_j M_1^{(j)}(F_{X_\alpha Y_\alpha})$$

where $M_1^{(j)}(F_{X_\alpha Y_\alpha}) \stackrel{d}{=} M_1(F_{X_\alpha Y_\alpha})$ are independent. Now from Proposition 3.1(iii) in James (18), one has

$$X_\alpha \stackrel{d}{=} M_1(F_{X_\alpha Y_\alpha}),$$

which completes the result.

In view of this result, we proceed to give an expression for the density of $X_{\alpha,k\alpha}\beta_{k\alpha,n-k\alpha}$. Define

$$\psi_j^{(n,k)}(w) = (n-1)\int_0^{1/w}(1-wx)^{n-2}\vartheta_j^{(n,k)}(x)dx$$

where $\vartheta_j^{(n,k)}(x)$ is given in (5.6).

Lemma 7.1. *For $n = 2,3,\ldots$ and $k = 1,2,\ldots,$ the density of $X_{\alpha,k\alpha}\beta_{k\alpha,n-k\alpha}$ in w is given by*

$$(n-1)\int_0^{1/w}(1-wt)^{n-2}\Delta_{k\alpha}(t|F_{X_{\alpha,k\alpha}})dt$$

which, from Lemma 5.1, can also be expressed as

$$\sum_{j=0}^{k} \binom{k}{j} \sin\left(\frac{\pi}{2}(k-j)\right) \psi_j^{(n,k)}(w).$$

Proof. Using standard arguments one can write the density of $X_{\alpha,k\alpha}\beta_{k\alpha,n-k\alpha}$ as

$$\frac{\Gamma(n)w^{k\alpha-1}}{\Gamma(k\alpha)\Gamma(n-k\alpha)} \int_w^\infty (1-w/y)^{n-k\alpha-1} y^{-k\alpha} f_{X_{\alpha,k\alpha}}(y)dy. \tag{7.1}$$

But it is not difficult to show that for any $\theta > -\alpha$,

$$f_{1/X_{\alpha,\theta}}(y) = y^{-\theta} f_{X_{\alpha,\theta}}(y).$$

Hence making the change $r = 1/y$, the expression in (7.1) is a constant multiplies

$$w^{k\alpha-1} \int_0^{1/w} (1-wr)^{n-k\alpha-1} f_{X_{\alpha,k\alpha}}(r)dr.$$

Now arguing as in the proof of Lemma 5.1, based on the definition of the density of $X_{\alpha,k\alpha}$ provided by (5.5), we will arrive at the integral

$$w^{-1} \int_{tw}^{1} (1-u)^{n-k\alpha-1} (u-tw)^{k\alpha-1} du$$

where t refers to the argument in $\Delta'_{k\alpha}(t)$. So, it follows that the density in (7.1) can be expressed as

$$\frac{1}{w} \int_0^{\frac{1}{w}} (1-wt)^{n-1} \Delta'_{k\alpha}(t)dt,$$

and the result follows.

This leads to another description of the quantities in Proposition 7.1.

Proposition 7.2. *Suppose that for each* $0 < \alpha < 1$,

$$\gamma_\alpha(t) = L_\alpha \mathbb{E}[g(S'_\alpha/t)] f_\alpha(t).$$

Then, the EPPF for $\mathrm{PK}(\rho_\alpha, \gamma_\alpha)$ *is such that,*

$$V_{n,k} = \frac{\alpha^{k-2}\Gamma(k)}{\Gamma(n)} L_\alpha \sum_{j=0}^{k} \binom{k}{j} \sin\left(\frac{\pi}{2}(k-j)\right) \zeta_{\alpha,j}^{(n,k)} \tag{7.2}$$

where

$$\zeta_{\alpha,j}^{(n,k)} = \frac{[\sin(\pi\alpha)]^{k-j}}{\pi} \int_0^\infty \mathbb{E}\left[g\left(\frac{\beta_{1,n-1}}{r^{1/\alpha}}\right)\right] \frac{(r\cos(\pi\alpha)+1)^j\, r^{(k-j)-1}}{[r^2+2r\cos(\alpha\pi)+1]^k} dr.$$

Proof. This follows from Lemma 7.1 and Proposition 7.1 and applying a change of variable.

7.1.1. Example: Mittag Leffler and generalized Mittag Leffler functions

This example is also influenced by some results in (18) and concerns the Mittag Leffler function and some of its generalizations. Recall that the Mittag Leffler function may be defined as

$$\mathrm{E}_{\alpha,1}(-\lambda) = \mathbb{E}[\mathrm{e}^{-\lambda S_\alpha^{-\alpha}}] = \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{\Gamma(1+l\alpha)} = \mathbb{E}[\mathrm{e}^{-\lambda^{1/\alpha} X_\alpha}].$$

Furthermore, there is an integral representation

$$\mathrm{E}_{\alpha,1}(-\lambda) = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{\mathrm{e}^{-\lambda^{1/\alpha} y} y^{\alpha-1}}{y^{2\alpha} + 2y^\alpha \cos(\pi\alpha) + 1} dy$$

which allows one to compute the Mittag Leffler function. Now, define

$$\mathrm{E}^{(k+1)}_{\alpha,1+k\alpha}(-\lambda) = \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} \frac{[k+1]_l}{\Gamma(1+k\alpha+l\alpha)}. \tag{7.3}$$

Theorem 7.1 in James (18) shows that

$$\mathrm{E}^{(k+1)}_{\alpha,1+k\alpha}(-\lambda) = \mathbb{E}[\mathrm{e}^{-\lambda^{1/\alpha} X_{\alpha,k\alpha}}]. \tag{7.4}$$

In this section set $L_{\alpha,\lambda} = 1/\mathrm{E}_{\alpha,1}(-\lambda)$.

Proposition 7.3. *Let, for $0 < \alpha < 1$, $(\gamma_{\alpha,\lambda} : \lambda > 0)$ denote the family of densities each defined as*

$$\gamma_{\alpha,\lambda}(t) = L_{\alpha,\lambda} e^{-\lambda/t^\alpha} f_\alpha(t).$$

Then, the EPPF's of the PK$(\rho_\alpha, \gamma_{\alpha,\lambda})$ family satisfy, for each fixed (α, λ),

(i) $V_{n,1} = \frac{1}{\Gamma(n)} L_{\alpha,\lambda} \mathbb{E}[\mathrm{E}_{\alpha,1}(-(\beta_{1,n-1})^\alpha \lambda)]$.

(ii) For $k = 1, 2, \ldots, n$,

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(k)}{\Gamma(n)} L_{\alpha,\lambda} \mathbb{E}[\mathrm{E}^{(k+1)}_{\alpha,1+k\alpha}(-(\beta_{k\alpha,n-k\alpha})^\alpha \lambda)].$$

(iii) Equivalently,

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(k)}{\Gamma(n-k)} L_{\alpha,\lambda} \int_0^1 \left[\int_{\mathcal{S}_k} \prod_{l=1}^{k} \mathrm{E}_{\alpha,1}(-p_l^\alpha b^\alpha \lambda) dp_l \right] b^{k-1} (1-b)^{n-k-1} db$$

where $\mathcal{S}_k = \{(p_1, \ldots, p_k) : 0 < \sum_{i=1}^k p_i \le 1\}$.

(iv) $V_{n,k}$ is described by Proposition 7.2 with

$$\zeta^{(n,k)}_{\alpha,j} = \frac{[\sin(\pi\alpha)]^{k-j}}{\pi} \int_0^\infty \mathbb{E}[e^{-\lambda^{1/\alpha} \beta_{1,n-1}/r^{1/\alpha}}] \frac{(r\cos(\pi\alpha)+1)^j \, r^{(k-j)-1}}{[r^2 + 2r\cos(\alpha\pi) + 1]^k} dr.$$

Proof. First note that the relevant g function is

$$g_\lambda(x) = \mathrm{e}^{-\lambda^{1/\alpha} x}.$$

Hence statement (i) follows from statement (i) in Proposition 7.1 and the representation of the Mittag Leffler function. Statement (ii) follows from (7.4) and Proposition 7.1. Statement (iii) follows from statement (iii) in Proposition 7.1 and statement (iv) follows from Proposition 7.2. Statement (iii) is also a special variation of statement (v) in Theorem 7.1 in (18).

7.2. Beta-Gamma Classes

The previous section identifies a large class of EPPF's that can be computed by using the random variables X_α and $X_{\alpha,k\alpha}$. Compared with common random variables that one encounters in standard probability textbooks, these random variables are rather exotic. In this section we show, also using results that appear in (18), that we may define a large, and clearly important, class where the EPPF's are expressible in terms of expectations depending only on beta and gamma random variables.

Recall the identity in (1.5), which again may be found in Pitman (40, section 4.2) and Perman, Pitman and Yor (35). Statement (iii) in Proposition 3.2 in James (18) uses the result to establish the identity

$$G^{1/\alpha}_{\frac{\theta+\alpha}{\alpha}} \stackrel{d}{=} \frac{G_{\theta+\alpha}}{S_{\alpha,\theta+\alpha}} \stackrel{d}{=} \frac{G_{\theta+1}}{S_{\alpha,\theta}} \tag{7.5}$$

for $\theta > -\alpha$. James (18, section 3.0.1) also explains how this amounts to a rephrasing of an otherwise equivalent result in Lemma 6 in Bertoin and Yor (3). Now let

$$1/\Sigma_{\alpha,\theta} = \mathbb{E}[g(G_{\theta/\alpha+1}\beta^{\alpha}_{\theta+\alpha,1-\alpha})].$$

Then (7.5) suggests the following class.

Proposition 7.4. *Suppose that for each* $0 < \alpha < 1$, *and* $\theta > -\alpha$,

$$\gamma_\alpha(t) = \Sigma_{\alpha,\theta}\mathbb{E}\left[g\left(\frac{G^{\alpha}_{\theta+\alpha}}{t^\alpha}\right)\right] f_{\alpha,\theta}(t).$$

That is,

$$h(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)}\Sigma_{\alpha,\theta}t^{-\theta}\mathbb{E}\left[g\left(\frac{G^{\alpha}_{\theta+\alpha}}{t^\alpha}\right)\right].$$

Then, $\mathrm{PK}(\rho_\alpha, \gamma_\alpha)$ *has the following properties.*

(i) For $n = 1, 2, \ldots,$

$$V_{n,1} = \frac{\Gamma(\theta+1)}{\Gamma(n+\theta)}\Sigma_{\alpha,\theta}\mathbb{E}[g(G_{\theta/\alpha+1}\beta^{\alpha}_{\theta+\alpha,n-\alpha})].$$

(ii) For $k = 1, 2, \ldots, n$,

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(\theta+1)\Gamma(\theta/\alpha+k)}{\Gamma(\theta/\alpha+1)\Gamma(n+\theta)}\Sigma_{\alpha,\theta}\mathbb{E}[g(G_{\theta/\alpha+k}\beta^{\alpha}_{\theta+\alpha,n-\alpha})].$$

Proof. It suffices to examine the quantity

$$\mathbb{E}[S^{-\theta}_{\alpha,k\alpha}\beta^{\theta}_{k\alpha,n-k\alpha}\, g(yS^{-\alpha}_{\alpha,k\alpha}\beta^{\alpha}_{k\alpha,n-k\alpha})]$$

for each k, g and fixed y. Now noting the form of the density of $S_{\alpha,\theta}$ and that of a beta random variable, it follows that the above expectation can be written as

$$\frac{\Gamma(n)\Gamma(\theta/\alpha+k)}{\Gamma(n+\theta)\Gamma(k)}\mathbb{E}[g(yS^{-\alpha}_{\alpha,\theta+k\alpha}\beta^{\alpha}_{\theta+k\alpha,n-k\alpha})].$$

Now replacing y with $G^{\alpha}_{\theta+\alpha}$ shows that

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(\theta+1)\Gamma(\theta/\alpha+k)}{\Gamma(\theta/\alpha+1)\Gamma(n+\theta)}\Sigma_{\alpha,\theta}\mathbb{E}[g(G^{\alpha}_{\theta+\alpha}S^{-\alpha}_{\alpha,\theta+k\alpha}\beta^{\alpha}_{\theta+k\alpha,n-k\alpha})].$$

But, using the calculus of beta and gamma random variables, it follows that since $\theta+\alpha \le \theta+k\alpha$,

$$\frac{G_{\theta+\alpha}}{S_{\alpha,\theta+k\alpha}} \stackrel{d}{=} \beta_{\theta+\alpha,(k-1)\alpha}\frac{G_{\theta+k\alpha}}{S_{\alpha,\theta+k\alpha}}.$$

Additionally,

$$\beta_{\theta+\alpha,(k-1)\alpha}\beta_{\theta+k\alpha,n-k\alpha} \stackrel{d}{=} \beta_{\theta+\alpha,n-\alpha}.$$

The result is concluded by applying the identity in (7.5).

Note that one can also work with the other equality in (7.5). We do not discuss that here.

7.2.1. Example: Hermite Type

Note that

$$\mathbb{E}[\mathrm{e}^{-G^{\alpha}_{\theta+\alpha}/t^{\alpha}}] = \frac{1}{\Gamma(\theta+\alpha)}\int_0^{\infty}\mathrm{e}^{-(\frac{x}{t})^{\alpha}}x^{\theta+\alpha-1}\mathrm{e}^{-x}dx$$

which looks like some sort of generalized Hermite function. We can use this fact along with Proposition 7.4 to calculate an EPPF, based on the mixing density, which we deliberately write in the less obvious form,

$$\gamma_{\alpha,\lambda}(t) = \frac{\Sigma_{\alpha,\theta}(\lambda)f_{\alpha,\theta}(t)}{\Gamma(\theta+\alpha)}\int_0^{\infty}\mathrm{e}^{-\lambda(\frac{x}{t})^{\alpha}}x^{\theta+\alpha-1}\mathrm{e}^{-x}dx \tag{7.6}$$

where

$$1/\Sigma_{\alpha,\theta}(\lambda) = \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)\Gamma(1-\alpha)}\int_0^1(1+\lambda u^{\alpha})^{-(\theta/\alpha+1)}u^{\theta+\alpha-1}(1-u)^{1-\alpha-1}du$$

which is the same as

$$\mathbb{E}[\mathrm{e}^{-\lambda G_{\theta/\alpha+1}\beta^{\alpha}_{\theta+\alpha,1-\alpha}}]$$

Here again

$$g(x) = \mathrm{e}^{-x},$$

which leads to the next result.

Proposition 7.5. *Let $\gamma_{\alpha,\lambda}$ be specified by (7.6) for all $0 < \alpha < 1$ and $\theta > -\alpha$. Then, the EPPF of* $\mathrm{PK}(\rho_\alpha, \gamma_{\alpha,\lambda})$ *is specified by*

$$V_{n,k} = \Phi^{(k)}_{\alpha,\theta}(\lambda) \int_0^1 (1+\lambda u^\alpha)^{-(\theta/\alpha+k)} u^{\theta+\alpha-1}(1-u)^{n-\alpha-1} du$$

where

$$\Phi^{(k)}_{\alpha,\theta}(\lambda) = \frac{\alpha^{k-1}\Gamma(\theta+1)\Gamma(\theta/\alpha+k)\Sigma_{\alpha,\theta}(\lambda)}{\Gamma(\theta/\alpha+1)\Gamma(\theta+\alpha)\Gamma(n-\alpha)}.$$

7.3. Comment on Composition Classes

Recall that if one evaluates a stable subordinator at a positive random time, say T, it is equivalent in distribution to

$$S_\alpha T^{1/\alpha}$$

having Laplace transform

$$\mathbb{E}[\mathrm{e}^{-S_\alpha T^{1/\alpha}}] = \mathrm{e}^{-\psi(\lambda^\alpha)}$$

where $\mathrm{e}^{-\psi(\lambda)}$ is the Laplace transform of T evaluated at λ. We see that the two classes in the previous sections are special cases of this where one chooses T in such a way where we can work more easily with the stable random variable. It is clear that many models may be derived using this general idea. That is, based on

$$h(t) = \mathbb{E}[g(tT^{1/\alpha})].$$

Furthermore, since g is arbitrary one can create quite a few variations of this.

8. EPPF's via ranked functionals of self-similar Markovian excursions

We close this paper by looking at a large class of mixing densities that probably would not immediately come to mind. This is partly because, as we shall show, the apparently complex form of them. The mixing distributions are based on the results of the interesting paper of Pitman and Yor (44). In fact, in the general case the densities do not have known closed forms, which means that $h(t)$ does not have an explicit form and hence brute force simulation methods or the calculus of H- or G-functions cannot be used directly. However, as we shall show, these are members of the Beta-Gamma class, and we will indeed be able to get explicit results by exploiting Corollary 7 in Pitman and Yor (44). For clarity, we start off with a simpler yet still challenging case.

8.1. Example: Hyperbolic Tangent and Kolmogorov's Formula

First note the famous random variable

$$M_1^{\mathrm{br}} := \max_{0\leq u\leq 1} |B_u^{\mathrm{br}}|$$

where B_u^{br} denotes standard Brownian bridge on $[0,1]$. One has the Kolmogorov's formula,

$$\mathbb{P}(M_1^{\mathrm{br}} \leq x) = \sum_{l=-\infty}^{\infty} (-1)^l e^{-2l^2x^2}.$$

Furthermore, one has the remarkable formula

$$\mathbb{P}(|B_1|M_1^{\mathrm{br}} \leq y) = \tanh(y), \tag{8.1}$$

where B_1 is a standard Gaussian random variable independent of B^{br}. See Pitman (40, p. 203) for this description. We will try to be quite transparent at this point. We want to use Proposition 7.4 to obtain nice EPPF's generated by a density related to M_1^{br}. The key to exploiting these results is that

$$|B_1|^2 \overset{d}{=} 2G_{1/2}.$$

Looking at Proposition 7.4, we want this to apply for all α, we solve the equation

$$\frac{\theta}{\alpha} + 1 = 1/2$$

which leads to

$$\theta = -\alpha/2,$$

which is less that 0 but greater than $-\alpha$, hence in the acceptable range of Proposition 7.4. This leads us to construct the following mixing density for each $\tau > 0$,

$$\gamma_{\alpha,\alpha/2,\tau}(t) = f_{\alpha,-\alpha/2}(t)\mathbb{P}(2(M_1^{\mathrm{br}})^2 \leq \tau t^{\alpha}/G_{\alpha/2}^{\alpha})\Sigma_{\alpha,\alpha/2}(\tau) \tag{8.2}$$

where by Proposition 7.4 and (8.1),

$$1/\Sigma_{\alpha,\alpha/2}(\tau) = \mathbb{P}(|B_1|M_1^{\mathrm{br}} \leq \sqrt{\tau}\beta_{\alpha/2,1-\alpha}^{-\alpha/2}) = \mathbb{E}[\tanh(\sqrt{\tau}\beta_{\alpha/2,1-\alpha}^{-\alpha/2})].$$

Now define probabilities

$$p_{n,k}(\tau|\alpha) = \mathbb{P}(2G_{k-1/2}(M_1^{\mathrm{br}})^2 \leq \beta_{\alpha/2,n-\alpha}^{-\alpha}\tau). \tag{8.3}$$

This leads to the following result.

Proposition 8.1. *Let for $0 < \alpha < 1$, and each $\tau > 0$, $\mathrm{PK}(\rho_\alpha, \gamma_{\alpha,\alpha/2,\tau})$ denote the Poisson Kingman distribution with mixing density specified in (8.2). In addition, let $p_{n,k}(\tau|\alpha)$ denote the probabilities defined in (8.3). Then, $\mathrm{PK}(\rho_\alpha, \gamma_{\alpha,\alpha/2,\tau})$ has the following properties.*

(i) For each n,

$$V_{n,1} = \frac{\Gamma(1-\alpha/2)}{\Gamma(n-\alpha/2)} \frac{\mathbb{E}[\tanh(\sqrt{\tau}\beta_{\alpha/2,n-\alpha}^{-\alpha/2})]}{\mathbb{E}[\tanh(\sqrt{\tau}\beta_{\alpha/2,1-\alpha}^{-\alpha/2})]}.$$

(ii) For $k = 1, 2, \ldots, n$,

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(1-\alpha/2)\Gamma(k-1/2)}{\Gamma(1/2)\Gamma(n-\alpha/2)} \frac{p_{n,k}(\tau|\alpha)}{\mathbb{E}[\tanh(\sqrt{\tau}\beta_{\alpha/2,1-\alpha}^{-\alpha/2})]}.$$

(iii) For $\tau > 0$, applying (6.2) yields the following recursion,

$$p_{n+1,k+1}(\tau|\alpha) = \frac{(n-\alpha/2)}{(k-1/2)\alpha}[p_{n,k}(\tau) - p_{n+1,k}(\tau)] + p_{n+1,k}(\tau).$$

Hence, these probabilities are completely determined by

$$\mathbb{E}[\tanh(\sqrt{\tau}\beta_{\alpha/2,n-\alpha}^{-\alpha/2})] = p_{n,1}(\tau).$$

Proof. For clarity, it follows from the form of the density in (8.2) that Proposition 7.4 applies with

$$g(x) = \mathbb{P}(2(M_1^{\mathrm{br}})^2 \le \tau/x)$$

and $\theta = -\alpha/2$. Furthermore,

$$\mathbb{E}[g(G_{k-1/2}\beta_{\alpha/2,n-\alpha}^{\alpha})] = p_{n,k}(\tau|\alpha)$$

which specializes in the case of $k = 1$ to

$$\mathbb{P}(|B_1|M_1^{\mathrm{br}} \le \sqrt{\tau}\beta_{\alpha/2,n-\alpha}^{-\alpha/2}) = \mathbb{E}[\tanh(\sqrt{\tau}\beta_{\alpha/2,n-\alpha}^{-\alpha/2})].$$

due to (8.1).

8.2. General result

Next we first sketch out some of the notation and definitions given in Pitman and Yor (44). We ask the interested to consult that work for more details. Note instead of γ and α, used in that work we use κ and δ. The first change is obviously to avoid confusion with the mixing density. The second as in the previous result is that δ will be associated with a stable subordinator, and other relevant quantities, not directly related to our use of α. We now paraphrase Pitman and Yor (44, p. 366-367) which describes the general random variables we are interested in.

In this section $B := (B_t, t > 0)$ denotes a real or vector valued β-self similar strong Markov process, with starting state 0 which is a recurrent point for B. Let $(e_t, 0 \le t \le V_e)$ denote a generic excursion path, where V_e is the *lifetime* or

length of e. Let F be a non-negative measurable functional of excursions e, and let $\kappa > 0$. Call F a *κ-homogeneous functional of excursions of B* if

$$F(e_t, 0 \le t \le V_e) = V_e^{\kappa} F(V_e^{-\beta} e_{uV_e}, 0 \le u \le 1). \tag{8.4}$$

In particular, we (Pitman and Yor) have in mind the following functionals F: length, maximum height, maximum absolute height, area, maximum local time. Now we restate their Theorem 3.

Theorem 8.1 (Pitman and Yor (44))**.** *Let F be a κ-homogeneous functional of excursions of B, let $F_* := F(B^{\mathrm{ex}})$ for a standard excursion B^{ex}, and suppose that $\mathbb{E}[(F_*)^{\delta/\kappa}] < \infty$, for $0 < \delta < 1$. Then the strictly positive values of $F(e)$ as e ranges over the countable collection of excursions of B^{br} can be arranged as a sequence*

$$F_1^{\mathrm{br}} \ge F_2^{\mathrm{br}} \ge \cdots > 0.$$

Let Γ_δ be a random variable, independent of B^{br}, with the gamma$(\delta, 1)$ density

$$\mathbb{P}(\Gamma_\delta \in dt)/dt = \Gamma(\delta)^{-1} t^{\delta-1} e^{-t} \qquad \textit{for } t > 0. \tag{8.5}$$

Fix $\lambda > 0$. Then, the joint distribution of the sequence $(F_j^{\mathrm{br}}, j = 1, 2, \ldots)$ is uniquely determined by the equality in distribution

$$\left(\mu(\lambda^{-\gamma}\Gamma_\delta^{\kappa} F_j^{\mathrm{br}}), j = 1, 2, \ldots\right) \stackrel{d}{=} (T_j^*, j = 1, 2, \ldots) \textit{ where } T_j^* := \sum_{i=1}^{j} \varepsilon_i/\varepsilon_0 \tag{8.6}$$

for independent standard exponential variables $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$, and μ is the function determined as follows by δ, λ, κ and the distribution of F_*:

$$\mu(x) := \int_0^\infty \frac{\delta\lambda^{-\delta}}{\Gamma(1-\delta)} t^{-\delta-1} e^{-\lambda t}\, \mathbb{P}(F_* > xt^{-\kappa})\, dt. \tag{8.7}$$

As noted by Pitman and Yor (44), the above result gives a characterization of the random variables F_j^{br} but it does not provide an explicit expression for their distribution, as for instance in the special case of M_1^{br}. Nonetheless, letting $\mathbb{P}_\delta$ denote the law of these F_j^{br}, depending on δ as described above, we can construct a rich class of EPPF's from these random variables. Specifically, Theorem 3 and Corollary 7 in Pitman and Yor (44) along with Proposition 7.4 suggests the following construction of mixing distributions.

First we solve the equation

$$\frac{\theta}{\alpha} + 1 = \delta$$

which leads to

$$-\alpha < \theta = (\delta - 1)\alpha < 0.$$

Now construct mixing densities as follows. For each α, j and δ,

$$\gamma_{\alpha,j,\delta}(t) = (\Sigma_j) \times \mathbb{P}_\delta\left(\lambda^{-\kappa} F_j^{\mathrm{br}} \ge w(t/G_{\delta\alpha})^{\alpha\kappa}\right) f_{\alpha,(\delta-1)\alpha}(t) \tag{8.8}$$

where

$$1/\Sigma_j = \mathbb{E}\left\{\left[\frac{\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,1-\alpha})}{1+\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,1-\alpha})}\right]^j\right\} = \mathbb{P}_\delta\left(\lambda^{-\kappa}G^\kappa_\delta F^{\mathrm{br}}_j \geq w\beta^{-\alpha\kappa}_{\alpha\delta,1-\alpha}\right)$$

and μ defined in (8.7) is a quite general function, some examples are given in Pitman and Yor (44). Now define

$$p_{n,k}(w|\delta,\alpha,j) = \mathbb{P}_\delta\left(\lambda^{-\kappa}G^\kappa_{k+\delta-1}F^{\mathrm{br}}_j \geq w\beta^{-\alpha\kappa}_{\alpha\delta,n-\alpha}\right). \tag{8.9}$$

The next result follows from the above discussion combined with Proposition 7.4 and a bit of algebra.

Theorem 8.2. *Let, for $0<\alpha<1$, $0<\delta<1$, $j=1,2,\ldots$, $\textsc{pk}(\rho_\alpha,\gamma_{\alpha,j,\delta})$ denote the Poisson Kingman distribution with mixing density specified in (8.8). In addition, let $p_{n,k}(w|\delta,\alpha,j)$ denote the probabilities defined in (8.9). Then, $\textsc{pk}(\rho_\alpha,\gamma_{\alpha,j,\delta})$ has the following properties.*

(i) For each n,

$$V_{n,1} = \frac{\Gamma(1-(1-\delta)\alpha)}{\Gamma(n-(1-\delta)\alpha)}\frac{\mathbb{E}\left\{\left[\frac{\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,n-\alpha})}{1+\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,n-\alpha})}\right]^j\right\}}{\mathbb{E}\left\{\left[\frac{\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,1-\alpha})}{1+\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,1-\alpha})}\right]^j\right\}}.$$

(ii) For $k=1,2,\ldots,n$,

$$V_{n,k} = \frac{\alpha^{k-1}\Gamma(1-(1-\delta)\alpha)\Gamma(k-(1-\delta))}{\Gamma(\delta)\Gamma(n-(1-\delta)\alpha)}\frac{p_{n,k}(w|\delta,\alpha,j)}{\mathbb{E}\left\{\left[\frac{\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,1-\alpha})}{1+\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,1-\alpha})}\right]^j\right\}}.$$

(iii) For $\lambda>0$, applying (6.2) yields the following recursion,

$$\begin{aligned} p_{n+1,k+1}(w|\alpha,\delta,j) \quad &= \quad \frac{(n-(1-\delta)\alpha)}{(k+\delta-1)\alpha}\left[p_{n,k}(w|\alpha,\delta,j)-p_{n+1,k}(w|\alpha,\delta,j)\right] \\ &\qquad + p_{n+1,k}(w|\alpha,\delta,j). \end{aligned}$$

Hence, these probabilities are completely determined by

$$\mathbb{E}\left\{\left[\frac{\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,n-\alpha})}{1+\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,n-\alpha})}\right]^j\right\} = p_{n,1}(w|\alpha,\delta,j).$$

Proof. The result again follows from Proposition 7.4. Now from (8.8), fixing j, α, δ and w,

$$g(x) = \mathbb{P}_\delta(\lambda^{-\kappa} F_j^{\mathrm{br}} \geq w x^{-\kappa}).$$

Furthermore, $\theta = (\delta - 1)\alpha$. Hence from Proposition 7.4,

$$\mathbb{E}[g(G_{k+\delta-1}\beta^{\alpha}_{\delta\alpha,n-\alpha})] = p_{n,k}(w|\delta,\alpha,j)$$

and when $k = 1$, one uses Corollary 7 or Theorem 3 in Pitman and Yor (44) to obtain

$$\mathbb{E}\left[g(G_\delta \beta^{\alpha}_{\delta\alpha,n-\alpha})\right] = \mathbb{E}\left\{\left[\frac{\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,n-\alpha})}{1+\mu(w\beta^{-\alpha\kappa}_{\alpha\delta,n-\alpha})}\right]^j\right\}.$$

8.2.1. Bessel Bridges

As an interesting special case, we now follow the example given in Pitman and Yor (44, p. 375). Let

$$h_{-\delta}(x) = \frac{I_{-\delta}(x)}{I_\delta(x)}$$

denote the ratio of two modified Bessel functions, which follows as a special case of (8.7). Then setting $\kappa = 1/2$, $\lambda = 1/2$, $F_j^{\mathrm{br}} \stackrel{d}{=} M_j^{\mathrm{br}}$, where under $\mathbb{P}_\delta$ are the ranked heights of excursion of a standard Bessel bridge of dimension $2 - 2\delta$. Hence, in this case,

$$g(x) = \mathbb{P}_\delta(\sqrt{2} M_j^{\mathrm{br}} \geq w x^{-1/2}),$$

and we will use, from Pitman and Yor (44, p. 375),

$$\mathbb{P}_\delta(\sqrt{2G_\delta} M_j^{\mathrm{br}} \geq w) = (1 - h_{-\delta}(w))^j.$$

Thus Theorem 8.2 specializes as follows.

Proposition 8.2. *Consider the setting in Theorem 8.2 with $\lambda = 1/2$ and $\kappa = 1/2$, such that for each j, $F_j^{br} \stackrel{d}{=} M_j^{br}$ are the ranked heights of excursion of a standard Bessel bridge of dimension $2 - 2\delta$. Then,*

$$V_{n,1} = \frac{\Gamma(1-(1-\delta)\alpha)}{\Gamma(n-(1-\delta)\alpha)} \frac{\mathbb{E}\left\{\left[1 - h_{-\delta}(w\beta^{-\alpha/2}_{\alpha\delta,n-\alpha})\right]^j\right\}}{\mathbb{E}\left\{\left[1 - h_{-\delta}(w\beta^{-\alpha/2}_{\alpha\delta,1-\alpha})\right]^j\right\}}$$

and the $V_{n,k}$ are obtained from $V_{n,1}$ via the recursion formula based on obvious adjustments in Theorem 8.2. This result generalizes Proposition 8.1.

9. Appendix: Fox H- and Meijer G-functions

Definition 9.1 (H-Function). *For integers m, n, p, q such that $0 \le m \le q, 0 \le n \le p$, for $a_i, b_j \in \mathbb{C}$ with $\mathbb{C}$, the set of complex numbers, and for $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$, $i = 1, 2, \ldots, p; j = 1, 2, \ldots, q$, the H-function (9) is defined via a Mellin-Barnes type integral (see (36; 1; 32; 27)) on the complex plane in the form*

$$\begin{aligned} H_{p,q}^{m,n}(z) &\equiv H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_i, \alpha_i)_1^p \\ \\ (b_j, \beta_j)_1^q \end{matrix} \right] \equiv H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right] \\ &= \frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)} z^{-s} ds. \end{aligned} \tag{9.1}$$

Here

$$z^{-s} = \exp\left[-s(\log|z| + i \arg z)\right], \qquad z \neq 0, \qquad i = \sqrt{-1},$$

where $\log|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value. An empty product in (9.1), if it occurs, is taken to one, and the poles

$$b_{jl} = \frac{-b_j - l}{\beta_j}, \quad j = 1, \ldots, m; l = 0, 1, 2, \ldots, \tag{9.2}$$

of the gamma functions $\Gamma(b_j + \beta_j s)$ and the poles

$$a_{ik} = \frac{-a_i - k}{\alpha_i}, \quad i = 1, \ldots, n; k = 0, 1, 2, \ldots, \tag{9.3}$$

of the gamma functions $\Gamma(a_i + \alpha_i s)$ do not coincide:

$$\alpha_i(b_j + l) \neq \beta_j(a_i - k - 1), \quad i = 1, \ldots, n; j = 1, \ldots, m; k, l = 0, 1, 2, \ldots.$$

$\mathfrak{L}$ in (9.1) is the infinite contour which separates all the poles b_{jl} in (9.2) to the left and all the poles a_{ik} in (9.3) to the right of $\mathfrak{L}$, and has one of the following forms:

- *(i) $\mathfrak{L} = \mathfrak{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;*
- *(ii) $\mathfrak{L} = \mathfrak{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;*
- *(iii) $\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$, where $\gamma \in \mathbb{R} = (-\infty, \infty)$.*

Refer to Kilbas and Saigo (20) for more discussion about H-functions, and in particular, Theorem 1.1 for the situation in which $H^{m,n}_{p,q}(z)$ defined by (9.1) makes sense.

Definition 9.2 (Meijer G-Function)**.** *In general the Meijer G-function (29; 30; 31) is defined by the following Mellin-Barnes type integral on the complex plane:*

$$\begin{aligned} G^{m,n}_{p,q}(z) &\equiv G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}(a_i)_1^p\\ \\ (b_j)_1^q\end{matrix}\right) \equiv G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ \\ b_1,\ldots,b_q\end{matrix}\right) \\ &= \frac{1}{2\pi i}\int_{\mathfrak{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{j=m+1}^{q}\Gamma(1-b_j-s)} z^{-s}ds, \end{aligned} \tag{9.4}$$

where the contour of integration $\mathfrak{L}$ is set up to lie between the poles of $\Gamma(a_i+s)$ and the poles of $\Gamma(b_j+s)$. The G-function is defined under the following hypothesis

- $0 \le m \le q, 0 \le n \le p$ *and* $p \le q-1$*;*
- $z \ne 0$*;*
- *no couple of b_j, $j = 1, 2, \ldots, m$ differs by an integer or a zero;*
- *the parameters $a_i \in \mathbb{C}$ and $b_j \in \mathbb{C}$ are so that no pole of $\Gamma(b_j+s), j = 1, 2, \ldots, m$ coincide with any pole of $\Gamma(a_i+s), i = 1, 2, \ldots, n$;*
- $a_i - b_j \ne 1, 2, 3, \ldots$ *for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$; and*
- *if $p = q$, then the definition makes sense only for $|z| < 1$.*

Refer to (6, Sec. 5) for a more thorough discussion of the G-function. Both H-functions and G-functions are very general functions whose special cases cover most of the mathematical functions such as the trigonometric functions, Bessel functions and generalized hypergeometric functions. Nonetheless, G-functions, but not H-functions, are implementable in *Mathematica* as `MeijerG[{a1,...,an},{a(n+1),...,ap},{b1,...` `{b(m+1),...,bq},z]`. Comparison between definitions (9.1) and (9.2) reveals that any G-function is an H-function, but not vice versa; when $\alpha_i = \beta_j = 1$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$,

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,1)_1^p\\ \\ (b_j,1)_1^q\end{matrix}\right] \equiv G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}(a_i)_1^p\\ \\ (b_j)_1^q\end{matrix}\right). \tag{9.5}$$

This implies that those H-functions which reduce to G-functions can be evaluated in *Mathematica*. Here are some results, available from Kilbas and Saigo (20, Section 2), that follow directly from the definition of the H-function in (9.1).

Property 9.1. The H-function is symmetric in the set of pairs $(a_1,\alpha_1),\ldots,(a_n,\alpha_n)$; in $(a_{n+1},\alpha_{n+1}),\ldots,(a_p,\alpha_p)$; in $(b_1,\beta_1),\ldots,(b_m,\beta_m)$ and in $(b_{m+1},\beta_{m+1}),\ldots,(b_q,\beta_q)$.

Property 9.2. If one of the pairs $(a_i, \alpha_i), i = 1, 2, \dots, n$, equals one of the pairs $(b_j, \beta_j), j = m+1, \dots, q$, and $n \geq 1, q > m$, then, for instance,

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^p\\ \\ (b_j,\beta_j)_1^{q-1},(a_1,\alpha_1)\end{matrix}\right] = H^{m,n-1}_{p-1,q-1}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_2^p\\ \\ (b_j,\beta_j)_1^{q-1}\end{matrix}\right] \tag{9.6}$$

or, if one of the pairs $(a_i, \alpha_i), i = n+1, \dots, p$, equals one of the pairs $(b_j, \beta_j), j = 1, 2, \dots, m$ and $m \geq 1, p > n$, then, for instance,

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^{p-1},(b_1,\beta_1)\\ \\ (b_j,\beta_j)_1^{q}\end{matrix}\right] = H^{m-1,n}_{p-1,q-1}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^{p-1}\\ \\ (b_j,\beta_j)_2^{q}\end{matrix}\right]. \tag{9.7}$$

Property 9.3. There holds the relation

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^p\\ \\ (b_j,\beta_j)_1^{q}\end{matrix}\right] = H^{n,m}_{q,p}\left[\frac{1}{z}\,\middle|\,\begin{matrix}(1-b_i,\beta_i)_1^q\\ \\ (1-a_j,\alpha_j)_1^{p}\end{matrix}\right]. \tag{9.8}$$

Property 9.4. For $y > 0$, there holds the relation

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^p\\ \\ (b_j,\beta_j)_1^{q}\end{matrix}\right] = yH^{m,n}_{p,q}\left[z^y\,\middle|\,\begin{matrix}(a_i,k\alpha_i)_1^p\\ \\ (b_j,k\beta_j)_1^{q}\end{matrix}\right]. \tag{9.9}$$

Property 9.5. For $\sigma \in \mathbb{C}$, there holds the relation

$$z^\sigma H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^p\\ \\ (b_j,\beta_j)_1^{q}\end{matrix}\right] = H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i+\sigma\alpha_i,\alpha_i)_1^p\\ \\ (b_j+\sigma\beta_j,\beta_j)_1^{q}\end{matrix}\right]. \tag{9.10}$$

Property 9.6. For $a, b \in \mathbb{C}$, there holds the relations

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a,0),(a_i,\alpha_i)_2^p\\ \\ (b_j,\beta_j)_1^{q}\end{matrix}\right] = \Gamma(1-a)H^{m,n-1}_{p-1,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_2^p\\ \\ (b_j,\beta_j)_1^{q}\end{matrix}\right] \tag{9.11}$$

when $\mathrm{Re}(1-a) > 0$ and $n \geq 1$;

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^p\\ \\ (b,0),(b_j,\beta_j)_2^{q}\end{matrix}\right] = \Gamma(b)H^{m-1,n}_{p,q-1}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^p\\ \\ (b_j,\beta_j)_2^{q}\end{matrix}\right] \tag{9.12}$$

when $\mathrm{Re}(b) > 0$ and $m \geq 1$;

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^{p-1},(a,0)\\ \\ (b_j,\beta_j)_1^{q}\end{matrix}\right] = \frac{1}{\Gamma(a)}H^{m,n}_{p-1,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^{p-1}\\ \\ (b_j,\beta_j)_1^{q}\end{matrix}\right] \tag{9.13}$$

when $\mathrm{Re}(a) > 0$ and $p > n$; and

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^p\\ \\(b_j,\beta_j)_1^{q-1},(b,0)\end{matrix}\right] = \frac{1}{\Gamma(1-b)} H^{m,n}_{p,q-1}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_1^p\\ \\(b_j,\beta_j)_1^{q-1}\end{matrix}\right] \tag{9.14}$$

when $\mathrm{Re}(1-b) > 0$ and $q > m$.

Property 9.7. *The following list shows how it is possible to express some mathematical functions in terms of the H-function:*

$$\begin{aligned}
e^z &= H^{1,0}_{0,1}\left[-z\,\middle|\,\begin{matrix}\text{---}\\ \\(0,1)\end{matrix}\right], \qquad \forall z\\
\cos z &= \frac{1}{\sqrt{\pi}} H^{1,0}_{0,2}\left[\frac{z^2}{4}\,\middle|\,\begin{matrix}\text{---}\\ \\(0,1),(\frac{1}{2},1)\end{matrix}\right], \qquad \forall z\\
\sin z &= \frac{2}{\sqrt{\pi}} H^{1,0}_{0,2}\left[\frac{z^2}{4}\,\middle|\,\begin{matrix}\text{---}\\ \\(0,1),(-\frac{1}{2},1)\end{matrix}\right], \qquad z \geq 0\\
\ln(1+z) &= H^{1,0}_{2,2}\left[z\,\middle|\,\begin{matrix}(1,1),(1,1)\\ \\(1,1),(0,1)\end{matrix}\right], \qquad |z| < 1\\
K_\eta(z) &= \frac{1}{2}\left(\frac{x}{2}\right)^{-a} H^{1,0}_{0,2}\left[\frac{z^2}{2}\,\middle|\,\begin{matrix}\text{---}\\ \\(\frac{a-\eta}{2},1),(\frac{a+\eta}{2},1)\end{matrix}\right], \qquad \forall z
\end{aligned} \tag{9.15}$$

where the last one is the modified Bessel function of the third kind.

Property 9.8. *The hypergeometric function, when $p \leq q$ or $p = q+1$ with $0 < |z| < 1$, can always be expressed in terms of the H-function:*

$$ {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right) = \frac{\prod_{i=1}^p \Gamma(a_i)}{\prod_{j=1}^q \Gamma(b_j)} H^{1,p}_{p,q+1}\left(-z\,\middle|\,\begin{matrix}(1-a_i,1)_1^p\\ \\(0,1),(1-b_j,1)_1^q\end{matrix}\right). \tag{9.16}$$

Refer to (20) for a more exhaustive list of relationships between the H-function and some other mathematical functions. With $\alpha_i = \beta_j = 1$, $i = 1,\ldots,p; j = 1,\ldots,q$ all the above properties for H-functions except Property 9.6 yield corresponding properties for G-functions due to the relationship (9.5). Before presenting two results from Prudnikov, Brychkov and Marichev (45, P.346,368) concerning calculations of integrals involving elementary functions and one or two G-functions that are used in this article, we define some more

notation as follows.

$$\begin{gathered} m,n,p,q,u,v,w,x = 0,1,2,\dots; r,s = 1,2,3,\dots \ (r,s \text{ are coprime});\\ 0 \le m \le q, \quad 0 \le n \le p, \quad 0 \le u \le x, \quad 0 \le v \le w;\\ b^* = u+v-\frac{w+x}{2}, \quad c^* = m+n-\frac{p+q}{2};\\ \mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}+1, \quad \rho = \sum_{j=1}^{x} d_j - \sum_{i=1}^{w} c_i + \frac{w-x}{2}+1;\\ \alpha,\sigma,\xi \in \mathbb{C}; \quad \sigma,\xi \ne 0. \end{gathered} \tag{9.17}$$

For any integer $\ell > 0$, define

$$\Delta(\ell, a) = \left(\frac{a}{\ell}, \frac{a+1}{\ell}, \dots, \frac{a+\ell-1}{\ell}\right), \tag{9.18}$$

and

$$\begin{aligned} \Delta(\ell,(a_i)_1^p) = \Bigg(&\frac{a_1}{\ell}, \frac{a_2}{\ell}, \dots, \frac{a_p}{\ell}, \frac{a_1+1}{\ell}, \frac{a_2+1}{\ell}, \dots, \frac{a_p+1}{\ell}, \dots,\\ &\frac{a_1+\ell-1}{\ell}, \frac{a_2+\ell-1}{\ell}, \dots, \frac{a_p+\ell-1}{\ell}\Bigg). \end{aligned} \tag{9.19}$$

Theorem 9.1 (Prudnikov, Brychkov and Marichev (45))**.** *Define notation in (9.17). If one of the 35 conditions stated in (45, P.346) holds, there holds the relation,*

$$\begin{aligned} &\int_0^\infty z^{\alpha-1} G_{w,x}^{u,v}\left(\sigma z \,\middle|\, \begin{matrix}(c_i)_1^w\\ \\ (d_i)_1^x\end{matrix}\right) G_{p,q}^{m,n}\left(\xi z^{s/r} \,\middle|\, \begin{matrix}(a_i)_1^p\\ \\ (b_i)_1^q\end{matrix}\right) dz\\ &\qquad = \frac{r^\mu s^{\rho+\alpha(x-w)-1}\sigma^{-\alpha}}{(2\pi)^{b^*(s-1)+c^*(r-1)}} G_{rp+sx,rq+sw}^{rm+sv,sn+su}\left(\frac{\xi^r r^{r(p-q)}}{\sigma^s s^{s(w-x)}} \,\middle|\, \begin{matrix}(A_i)_1^{rp+sx}\\ \\ (B_j)_1^{rq+sw}\end{matrix}\right) \end{aligned} \tag{9.20}$$

where

$$\begin{aligned} (A_i)_1^{rp+sx} \quad &= \quad (\Delta(r,a_1),\dots,\Delta(r,a_n),\Delta(s,1-\alpha-d_1),\dots,\\ &\qquad \Delta(s,1-\alpha-d_x),\Delta(r,a_{n+1}),\dots,\Delta(r,a_p)) \end{aligned}$$

and

$$\begin{aligned} (B_j)_1^{rq+sw} \quad &= \quad (\Delta(r,b_1),\dots,\Delta(r,b_m),\Delta(s,1-\alpha-c_1),\dots,\\ &\qquad \Delta(s,1-\alpha-c_w),\Delta(r,b_{m+1}),\dots,\Delta(r,b_q)). \end{aligned}$$

Corollary 9.1 (Prudnikov, Brychkov and Marichev (45))**.** *When $\sigma = y^{-1}$, $w,x,u = 1$, $v = d_1 = 0$, $c_1 = \beta$ in Theorem 9.1, (9.20) reduces to*

$$\begin{aligned} &\int_0^y z^{\alpha-1}(y-z)^{\beta-1} G_{p,q}^{m,n}\left(\xi z^{s/r} \,\middle|\, \begin{matrix}(a_i)_1^p\\ \\ (b_i)_1^q\end{matrix}\right) dz\\ &\quad = \frac{r^\mu s^{-\beta}\Gamma(\beta)}{(2\pi)^{c^*(r-1)} y^{1-\alpha-\beta}} G_{rp+s,rq+s}^{rm,rn+s}\left(\frac{\xi^r a^s}{r^{r(q-p)}} \,\middle|\, \begin{matrix}\Delta(s,1-\alpha),\Delta(r,(a_i)_1^p)\\ \\ \Delta(r,(b_j)_1^q),\Delta(s,1-\alpha-\beta)\end{matrix}\right). \end{aligned}$$

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